**J. T. S. Ma**
**P. K. C. Wang**

# Effect of Initial Air Content on the Dynamics of Bubbles in Liquids

Initiation and collapse of cavitation bubbles are strongly influenced by the gas content of the liquid. Degassed liquids can withstand large tensions without bubble formation: Harvey[1] subjected air-saturated water to pressures of $10^4$ psi for several minutes, forcing the air nuclei into solution; upon depressurization, the solution did not cavitate under conditions which produced cavitation before pressurization. In this note, we analyze the effect of initial air partial pressure on the growth rate and dynamic stability of a bubble.

We assume the bubble is spherical throughout its growth; this is valid if the radial acceleration and velocity are sufficiently small. We neglect air diffusion across the bubble wall, which Epstein and Plesset[2] showed is so slow, compared with the growth rate of bubbles, that it does not affect the bubble air content. The question of nucleation is avoided by assuming a finite initial radius. Thermal effects of evaporation and condensation of liquid vapor are neglected; heat flow is not important when the temperature is appreciably below the boiling point of the liquid. The liquid viscosity is neglected.

From Rayleigh's[3] theory, the bubble radius $R$ satisfies the equation

$$R d^2R/dt^2 + (3/2)(dR/dt)^2 = (1/\rho)[P_v - P_\infty + P_a - (2\sigma/R)] , \tag{1}$$

where $P_v$ is the vapor pressure of the liquid, $P_a$ is the partial pressure of air in the bubble, $P_\infty$ is the atmospheric pressure, $\sigma$ is the surface tension, and $t$ is time. For isothermal growth,

$$u d^2u/d\tau^2 + (3/2)(du/d\tau)^2 = [1 + \beta/u^3 - (1+\beta)/u]/(1+\beta) , \tag{2}$$

where

$$u = R/R_0 , \qquad \tau = (\Delta P + P_{ai}/\rho)^{1/2}(t/R_0) ,$$

$$\Delta P = P_v - P_\infty , \qquad \beta = P_{ai}/\Delta P ,$$

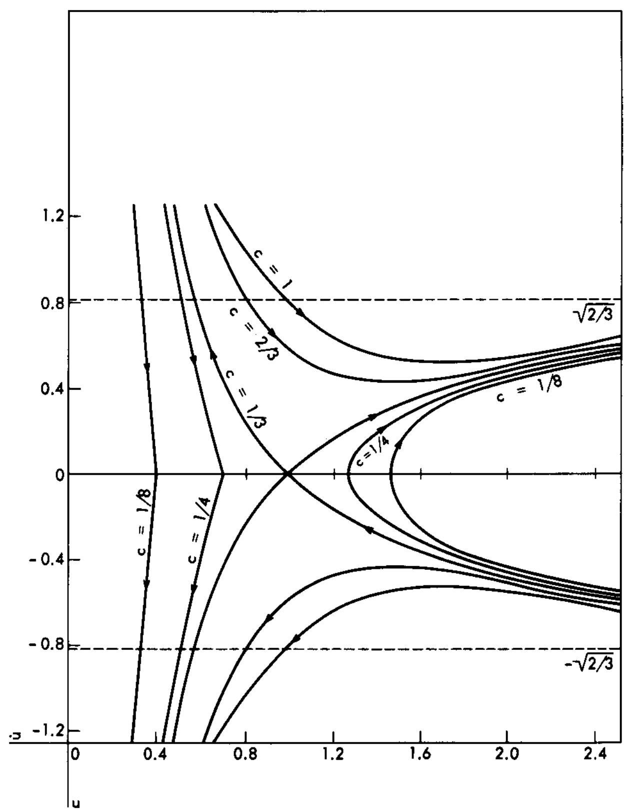
with $R_0$ the equilibrium radius $2\sigma/(\Delta P + P_{ai})$ and $P_{ai}$ the initial air pressure in the bubble (zero for a vapor bubble). The first integral of (2) is

$$du/d\tau = \pm\{[2/3 + 2\beta(\ln u)/u^3 - (1+\beta)/u + c/u^3]/(1+\beta)\}^{1/2} , \tag{3}$$

in which $c = (1+\beta)u_i^2(1 + u_i\dot{u}_i^2) - (2/3)u_i^3 - 2\beta \ln u_i$, where $u_i$ and $\dot{u}_i$ are the initial values of $u$ and $du/d\tau$ respectively. For $\beta = 0$,

$$du/d\tau = \pm(2/3 - 1/u + c'/u^3)^{1/2} , \tag{4}$$

*Figure 1* **$\dot{u}$ as a function of $u$ for $\beta = 0$ and various values of c.**



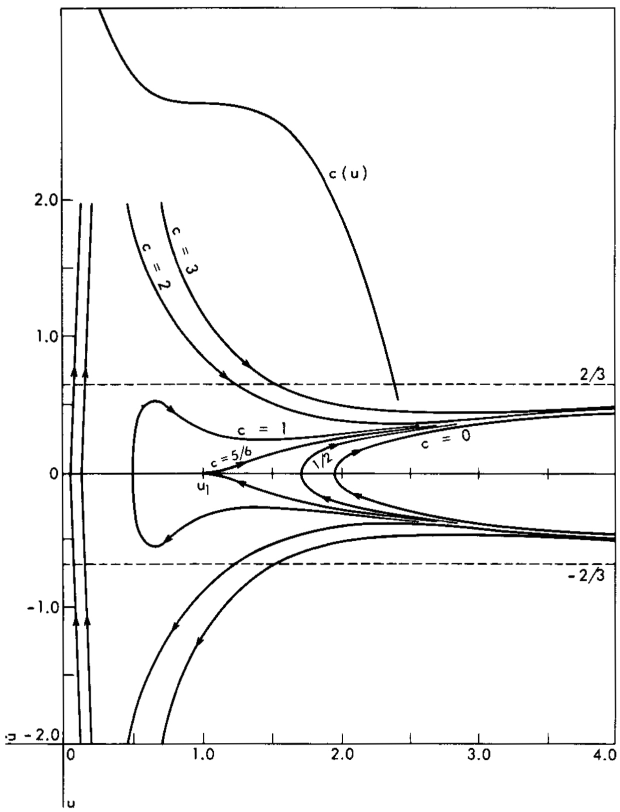



*Figure 2* **$\dot{u}$ as a function of $u$ for $\beta = 1/2$ and various values of c.**

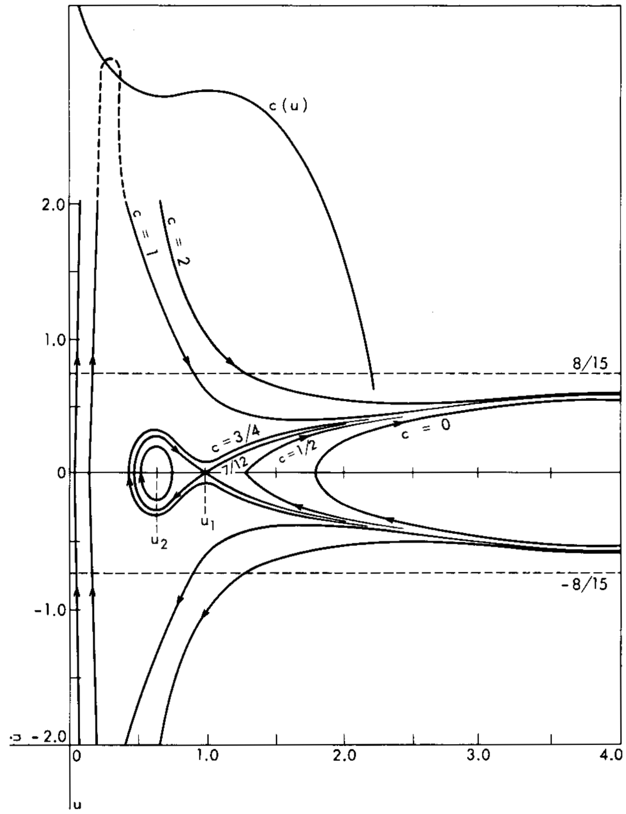



*Figure 3* **$\dot{u}$ as a function of $u$ for $\beta = 1/4$ and various values of c.**

where $c' = u_i^2[1 + u_i\dot{u}_i^2 - (2/3)u_i]$. This case was considered by Dergarabedian[4]; its phase plane trajectories are shown in Fig. 1.

Equation (2) expressed in first-order differential form is

$$d\dot{u}/du = \{[u^3 - (1+\beta)u^2 + \beta]/(1+\beta) - (3/2)u^3\dot{u}^2\}/u^4\dot{u}\ , \qquad (5)$$

where $\dot{u}$ denotes $du/d\tau$. The singularities are the points $(u_i, 0)$, with $u_1$, $u_2$, $u_3$ the roots of $u^3 - (1+\beta)u^2 + \beta = 0$, i.e., $u_1 = 1$, $u_{2,3} = (\beta/2)[1 \pm (1 + 4/\beta)^{1/2}]$. For $\beta = 0$, the only root is $u = 1$. The point $u = 1$, $\dot{u}_1 = 0$ on Fig. 1 is a singular point representing dynamically unstable equilibrium. For $\beta \neq 0$, the roots $u_1$ and $u_2$ represent, when positive, dynamic equilibrium points. The singularity at $u = 1$ corresponds to $R = R_0$ and the parameter $\beta$ determines the effect of the initial air on the other equilibrium radius. There are three possible types of dynamic behavior: if $\beta = 1/2$, $u_1 = u_2 = 1$; for any positive value of $\beta$ other than 1/2 there is another positive root $u_2$, $u_1 < u_2$ for $\beta > 1/2$ and $u_1 > u_2$ for $\beta < 1/2$; if $\Delta p$ is negative, so that $\beta < 0$, there is only one positive root $u = 1$, the other roots have no physical meaning. If $\beta = 1/2$, the two singularities coincide. The energy curve $c(u)$ is given by

$$c(u) = -(2/3)u^3 - 2\beta\ ln\ u + (1+\beta)u^2 + c\ , \qquad (6)$$

which has a point of inflection with a horizontal tangent at $u = 1$. Trajectories through $(1, 0)$ cusp at this point; there are no closed trajectories, hence no periodic solutions. Figure 2 shows the curve $c(u)$, and the trajectories of (3). For a particular pair of initial values $(u_i, \dot{u}_i)$, the trajectory is uniquely determined. Trajectories for positive and negative $\dot{u}$ correspond to growth and collapse, respectively. The singular point $u = 1$ is dynamically unstable.

For $\beta = 0$, the criterion $\dot{u} = 0$, $u > 1$ indicated growth. However, bubbles containing air tend to grow upon disturbance: for $u > 1$, the trajectories resemble those for $\beta = 0$, but for $u < 1$, the bubble is growing, not collapsing. Physically, when $u < 1$, the internal potential energy of the air causes the bubble to rebound before it shrinks to zero radius. For $0 < \beta \neq 1/2$, there are two singularities. Since the two cases are similar except that the singularities are interchanged, we discuss only the case $\beta < 1/2$ (Figure 3 corresponds to $\beta = 1/4$). The trajectories are closed curves surrounding the center point $(u_2, 0)$ unless the energy constant is large enough that the trajectory passes through the saddle point $(u_1, 0)$. For energies quite different from $c(u_2)$, there are two distinct types of open solution

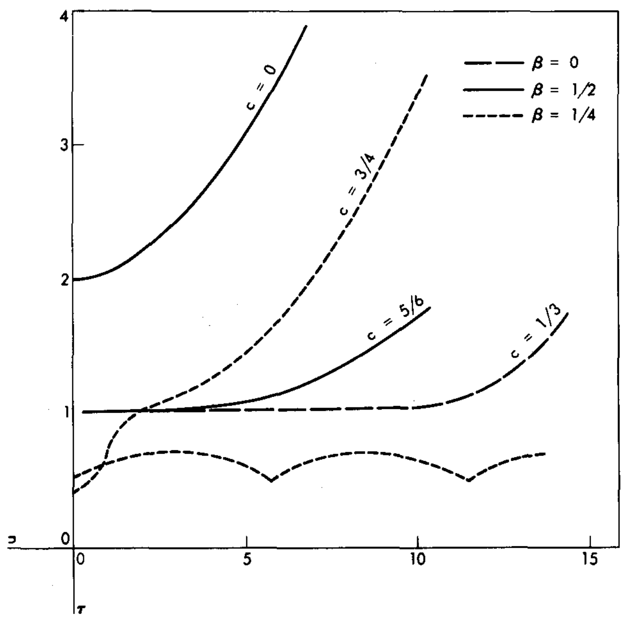



*Figure 4* **u as a function of τ.**

curves, i.e., two different types of aperiodic motion. The motion is periodic only for a small range of initial conditions.

The case $\beta < 0$ arises only if $\Delta P < 0$, since $P_{ai} \geqq 0$. For the only root $u_i$ to have physical meaning, $P_{ai}$ must exceed $|\Delta p|$: otherwise, the equilibrium radius, which is also the initial radius, would be negative.

To determine the dynamic equilibrium condition from (2), let

$$f(u) = 1 + \beta/u^3 - (1 + \beta)/u \,, \tag{7}$$

whence

$$[df/du]_{u=1} = 1 - 2\beta \,. \tag{8}$$

The range of non-negative $\beta$ for which $[df/du]_{u=1} \geqq 0$ is $1/2 \geqq \beta \geqq 0$, i.e., $4\sigma/3\Delta P \leqq R_0 \leqq 2\sigma/\Delta P$ with $\Delta P/2 \geqq P_{ai} \geqq 0$. Within this regime, the bubble is dynamically unstable; for $\beta = 1/2$, $df/du = 0$, it is neutrally stable. Equation (8) also shows that, for $u < 1$ (i.e., $R < R_0$), $df/du < 0$ for all $\beta > 1/2$, i.e., a bubble of radius $R < 4\sigma/3\Delta P$ is dynamically stable when $\Delta P > 0$. However, such a bubble would soon dissolve.

Figure 4 shows response curves for the cases $\beta = 0$, 1/4, 1/2 with different initial values of $c$. The integral form of (4) for $\beta = 0$ is

$$\tau - \tau_i = \int_{u_i}^{u} (2/3 - 1/\zeta + c'/\zeta^3)^{-1/2} d\zeta \,. \tag{9}$$

For $c' = 1/3$ (i.e., $u_1 = 1$, $\dot{u} = 0$), this can be integrated,[4] showing that the closer $u_1$ is to unity, the longer it takes for the bubble to grow. For $\beta \neq 0$, (3) shows that as $u$ increases without bound the asymptotic growth rate is linear in time; physically, *the effect of air is important in the initial growth, but, except for determining the asymptotic slope of the $u - \tau$ curve, it may not affect significantly the subsequent growth behavior.*

Figure 4 illustrates, for $\beta = 1/4$, the periodic oscillation when $u$ lies between the limits of the loop through the saddle point. Evidence of oscillating bubbles is found in hydraulic machinery and in ultrasonic cavitation. At the center point $u_2$, the bubble is in stable dynamic equilibrium. For values of $u$ within the loop, the trajectories are limit cycles, representing periodic solutions.

## References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