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Cubic Splines with Infinite Derivatives at Some Knots

Abstract: A generalization of cubic spline interpolation with vertical slopes at some knots is proposed. An existence theorem including an algorithm for constructing such generalized splines is proved. The resulting splines are obtained in closed form and they are partition invariant.

Introduction

Typically in industry (such as aerospace, shipbuilding and automotive), surfaces are designed in portions – often called "patches" – joined via some smoothness constraints. Each patch is defined in terms of functions belonging to an "allowable class." The class is chosen on the basis of criteria deemed desirable by the designer (a most important criterion being the kind of patches that can be manufactured with the available equipment). Spline functions [1], because of their desirable properties, are extensively used in surface design and other applications including the one that motivated this note. Here, an extension of their applicability is offered.

The design of surfaces having infinite derivatives (in one or more variables) at some points causes difficulties. Such is the case, for example, in the aerospace industry in the design of nose cones, nacelles, airfoils etc., a problem referred to in a recent article [2] and elsewhere. In our case, the problem arose in some work in mathematical biology [3] where surfaces of revolution are generated by the rotation of planar cubic splines having infinite derivatives at some points.

Although methods exist for coping with this problem, an alternate, more advantageous approach is given which is a natural extension of a very successful existing method [4, 5]. The resulting splines are obtained in closed form for either finite or infinite derivatives occurring within a given interval, where previously in the case of infinite derivatives an iterative process was used. This improves the computational efficiency of the existing algorithm and enhances the control of the surface design. Additionally, the splines obtained are *partition invariant.* This means that local (on sub-intervals) alterations of the spline can be made that do not change the spline elsewhere. When infinite derivatives occur, the splines obtained with previous methods are not partition invariant.

Formulation

We adapt Dimsdale's method [4, 5] of spline definition. Though the discussion is limited to two dimensions, the $x - w$ plane, it seems that this method can be generalized to three dimensions by replacing cubics with bicubics and ordinary with partial derivatives.

Let the partition $a = x_1 < x_2 < \cdots < x_N = b$ and the points (x_i, w_i) be given as in Fig. 1. In the parlance of surface designers the given points are called *knots*, but other often used terms are *joints* and *junction points.* It is desired to construct a spline – generalized in a natural way – passing through the knots and having infinite derivatives at x_{i_j}, $j = 1, \cdots, k \le N$. In Fig. 1 for example $k = 3$ with $i_1 = 1$, $i_2 = N - 2$ and $i_3 = N - 1$.

In the following we use the usual notation wherever possible. Specifically, R is the set of real numbers. $C(a, b)$ is the set of continuous functions $f: (a, b) \rightarrow R$, and the left and right side limits of the function values at b and a respectively are indicated by $f_-(b)$ and $f_+(a)$. We call $I_i = [x_i, x_{i+1}]$ for $i = 1, \cdots, N - 1$. One can write I_i' for an interval that has an x_{i_j} as an endpoint. Therefore, either $I_i' = I_{i_j}$ or $I_i' = I_{i_j-1}$, the later case is for $i_j > 1$. Finally we define $[a, b]' = [a, b] - \cup_{j=1}^{k}\{\partial I_{i_j-1} \cup \partial I_{i_j}\}$ (that is $[a, b]'$ is $[a, b]$ with all endpoints of I_i' excluded).

The generalization as well as the main result is included in the existence theorem below. In the next section a proof containing the algorithm for constructing the spline is provided, the result pertains to the knots at x_{i_j} – having infinite derivatives – satisfying the condition (*) below. That is letting $k_0 = k_{N-1} = k_N = 1$, $H_0 = H_{N-1} = 2$, $w_0 = w_1$, $w_{N+1} = w_N$, $k_i = (x_{i+1} - x_i)/(x_{i+2} - x_{i+1})$, $H_i = 2\,k_i(k_i + 1)$, $K_i = k_i^2 k_{i+1}$

$$R_i = 3[k_i^2(w_{i+2} - w_{i+1}) + w_{i+1} - w_i],\ i = 0, \cdots, N - 1$$

and

$$\Delta = \begin{bmatrix} H_0 & K_0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & H_1 & K_1 & 0 & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & 0 \\ \cdot & & & \cdot & \cdot & \cdot & K_{N-2} \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & H_{N-1} \end{bmatrix}, \quad R = \begin{bmatrix} R_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ R_{N-1} \end{bmatrix},$$

it is required that

$$(*) \qquad |\Delta_j| \neq 0 \ \forall_j = 1, \cdots, k$$

where $|\Delta_j|$ is the determinant of the matrix Δ_j obtained from Δ by replacing the jth column of Δ by the column vector R.

Theorem. $\exists s = s(x): [a, b] \rightarrow [a, b]$ and $W = W(s): [a, b] \rightarrow R \ni$

1. s is a monotone increasing and onto homeomorphism with

   $x(s) = s$ on $I_i \neq I_i'$

   $x(s)$ is a cubic polynomial on I_i'

2. $W(s)$ is a cubic polynomial on I_i, $\forall i = 1, \cdots, N-1$,
3. $W \in \tilde{C}^2 = \{\mathrm{f}: [a, b] \rightarrow R | \mathrm{f}'' \in C(a, b),$

   $$\mathrm{f}_+''(a) = \mathrm{f}_-''(b) = 0, \mathrm{f}(x_i) = w_i\},$$

4. W minimizes $\int_a^b \left(\frac{d^2\mathrm{f}}{ds^2}\right)^2 ds \ \forall \mathrm{f} \in \tilde{C}^2$ and
5. For $w(x) = W(s(x))$ and the knots at $x_{i_j}, j = 1, \cdots, k$ satisfying $(*)$,

   $$w \in \overline{C}^2 = \left\{\mathrm{f}: [a, b] \rightarrow R | \mathrm{f}'' \in C[a, b]', \ \mathrm{f}(x_i) = w_i, \lim_{x \rightarrow x_{i_j}} \frac{d\mathrm{f}}{dx} = \pm\infty\right\}.$$

The topology of the real line is preserved by the mapping s. In particular, closed sub-intervals are mapped into closed sub-intervals. We call $w = w(x)$ a *homeomorphic spline* in the $x - w$ plane, since it is obtained via the homeomorphism s, to distinguish it from $W = W(s)$ which is a spline in the $s - W$ plane. If there are no infinite derivative constraints, $w = w(x)$ is also a spline in the $x - w$ plane identical to the one obtained with Dimsdale's algorithm.

Algorithm and proof

The plan for constructing the homeomorphism s is shown in Fig. 2, where a piecewise intermediate parameterization $x \rightarrow u_i \rightarrow s$ for $x, s \in I_i$ is indicated.

To express a cubic polynomial in $u \in [0, 1]$ we employ the *Hermite basis*,

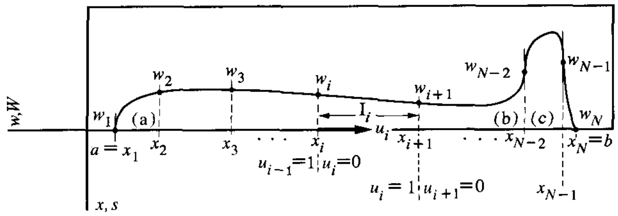

Figure 1 Desired curve passing through all knots and having vertical slopes at x_1, x_{N-2} and x_{N-1}.

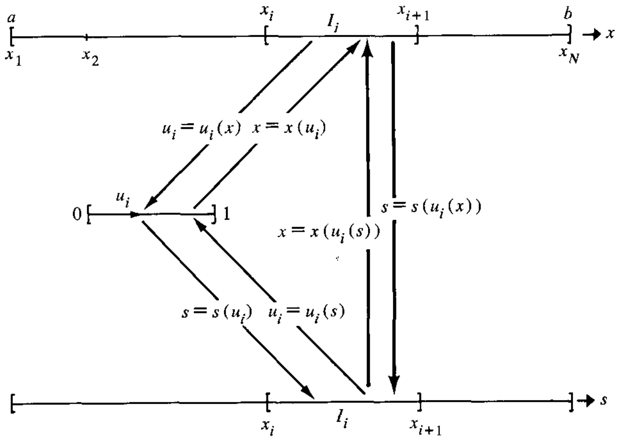

Figure 2 Construction of the parametrization s.

$$\alpha_0(u) = 2u^3 - 3u^2 + 1,$$

$$\alpha_1(u) = -2u^3 + 3u^2,$$

$$\alpha_2(u) = u^3 - 2u^2 + u,$$

$$\alpha_3(u) = u^3 - u^2;$$

whose convenience is indicated by the values in Table 1.

Let $A_i = x_{i+1} - x_i$ be the length of the interval I_i and $k_i = A_i / A_{i+1}, i = 1, \cdots, N-2, k_{N-1} = 1$ the ratio of lengths of consecutive intervals. On each interval I_i we define implicitly the *local parameter* $u_i \in [0, 1]$ for $x(u_i) \in I_i$ by

$$x(u_i) = x_i\alpha_0(u_i) + x_{i+1}\alpha_1(u_i) + B_i\alpha_2(u_i) + D_i\alpha_3(u_i). \quad (1)$$

Simultaneously, we define the cubic polynomial

$$W_i(u_i) = w_i\alpha_0(u_i) + w_{i+1}\alpha_1(u_i) + C_i\alpha_2(u_i) + k_iC_{i+1}\alpha_3(u_i). \quad (2)$$

The constants B_i, C_i and D_i will be determined so as to satisfy the assertions of the theorem. Note that $x(0) = x_i$, $x(1) = x_{i+1}$, $W_i(0) = w_i$ and $W_i(1) = w_{i+1}$. To avoid cumbersome notation we write u for the local parameter whenever the index i is unimportant or clear from the context.

Table 1 Hermite basis values.

j	$\alpha_j(0)$	$\alpha_j(1)$	$\alpha_j'(0)$	$\alpha_j'(1)$
0	1	0	0	0
1	0	1	0	0
2	0	0	1	0
3	0	0	0	1

The parametrization (1) allows the occurrence of infinite derivatives at a knot. That is,

$$\frac{dW_i(x_i)}{dx} = \frac{C_i}{B_i} \text{ and } \frac{dW_i(x_{i+1})}{dx} = \frac{k_i C_{i+1}}{D_i}.$$

Hence, for $C_i \neq 0$ (or $C_{i+1} \neq 0$) the derivative becomes infinite only when $B_i = 0$ (or $D_i = 0$). However, since

$$\frac{dW_i(0)}{du_i} = C_i \text{ and } \frac{dW_i(1)}{du_i} = k_i C_{i+1}$$

the derivative with respect to the local parameters is always finite.

• *Determination of* $s = s(x)$

The B_i and D_i are determined by imposing:

$$\lim_{x \to x_i^-} \frac{dW_{i-1}}{dx} = \lim_{x \to x_i^+} \frac{dW_i}{dx}, \tag{3}$$

which results in first order continuity of $W_i(x)$ at the knots with finite derivatives, and

$$x'(u) = 3u^2(-2A_i + B_i + D_i) + 2u(3A_i - 2B_i - D_i) + B_i \geq 0 \tag{4}$$

stemming from the requirement that the parametrization be one-to-one and orientation preserving. We distinguish two cases:

Finite derivatives at endpoints of I_i.

The simplest conditions for the positivity of the derivative are

$$-2A_i + B_i + D_i = 0$$

$$3A_i - 2B_i - D_i = 0$$

with $B_i > 0$

yielding $B_i = D_i = A_i > 0$ and the linear parametrization (with $x'(u_i) \geq 0$)

$$u_i = x - x_i / A_i \to u_i \in [0, 1] \text{ when } x \in I_i. \tag{5}$$

Infinite derivatives at endpoints of I_i.

Case a. The derivative is infinite at x_i *and finite at* x_{i+1}. This condition implies that $B_i = 0$ and $D_i \neq 0$. From (3) applied to x_{i+1}

$$\lim_{x \to x_{i+1}^-} \frac{dW_i}{dx} = \lim_{u_i \to 1} \frac{dW_i}{du_i}\frac{du_i}{dx}$$

$$= \lim_{u \to 1} \left[\frac{6w_i u(u-1) - 6w_{i+1}u(u-1)}{u[3u(-2A_i + D_i) + 2(3A_i - D_i)]} + \frac{C_i(3u-1)(u-1) + k_i C_{i+1} u(3u-2)}{u[3u(-2A_i + D_i) + 2(3A_i - D_i)]}\right]$$

$$= \frac{k_i C_{i+1}}{D_i} = \lim_{x \to x_{i+1}^+} \frac{dW_{i+1}}{dx} = \frac{C_{i+1}}{A_{i+1}} \tag{6}$$

where in the last step it was assumed that the derivative at x_{i+2} is finite so that (5) prevails on I_{i+1}. (The case of infinite derivative at x_{i+2} is settled in *b*.) Hence, $D_i = A_i$

$$x'(u) = A_i u(-3u + 4) > 0 \text{ for } u \in [0, 1]$$

and $x(u) = -A_i u^3 + 2A_i u^2 + x_i$, which upon rearrangement is

$$u^3 - 2u^2 + v_i = 0 \text{ with } v_i = \frac{x - x_i}{A_i}. \tag{7}$$

For $0 \leq v_i \leq 1$, (7) is an irreducible cubic whose three distinct real roots are given by [6],

$$u = \tfrac{2}{3} + \tfrac{4}{3}\cos\left[\tfrac{1}{3}\cos^{-1}\left(1 - \frac{3^3}{2^4}v_i\right)\right]. \tag{8}$$

At the endpoints of I_i we have, directly from (7),

$v_i = 0$ $u = 0$ (double root), $u = 2$ and

$$v_i = 1 \quad u = 1,\ u = \frac{1 \pm \sqrt{5}}{2}.$$

We seek then a monotone increasing portion of (8) such that $u(0) = 0$ and $u(1) = 1$. Such a branch exists, framed portion of Fig. 3- Case a, and is given by

$$u_i = \tfrac{2}{3} + \tfrac{4}{3}\cos\left[\frac{4\pi}{3} + \tfrac{1}{3}\operatorname{Cos}^{-1}\left(1 - \frac{3^3}{2^4}v_i\right)\right] \quad v_i \in [0, 1], \tag{9}$$

with the restriction $0 \leq \operatorname{Cos}^{-1} y \leq \pi$. We have from (9) that $u_i \in [0, 1]$ when $x \in I_i$.

Case b. The derivative is finite at x_i and infinite at x_{i+1}.

Here $B_i \neq 0$ and $D_i = 0$.

Applying (3) to x_i,

$$\lim_{x \to x_i^-} \frac{dW_{i-1}}{dx} = \frac{C_i}{A_i} = \lim_{x \to x_i^+} \frac{dW_i}{dx} = \lim_{u_i \to 0} \frac{dW_i}{du_i}\frac{du_i}{dx}$$

$$= \lim_{u \to 0} \left[\frac{6w_i u(u-1) - 6w_{i+1}u(u-1)}{3u^2(-2A_i + B_i) + 2u(3A_i - 2B_i) + B_i} + \frac{C_i(3u-1)(u-1) + k_i C_{i+1} u(3u-2)}{3u^2(-2A_i + B_i) + 2u(3A_i - 2B_i) + B_i}\right] = \frac{C_i}{B_i}, \tag{10}$$

where in the first step it was assumed that the derivative at x_{i-1} is finite so that (5) is valid on I_{i-1}. We find that

$B_i = A_i$, $x'(u) = A_i(-3u^2 + 2u + 1) \geq 0$ for $u \in [0, 1]$ with $x'(u) = 0 \rightarrow u = 1$, and

$$u^3 - u^2 - u + v_i = 0 \text{ where } v_i \text{ is as above.} \qquad (11)$$

As for (9), the inversion of (11) yields

$$u_i = \tfrac{1}{3} + \tfrac{4}{3}\cos\left[\frac{4\pi}{3} + \tfrac{1}{3}\operatorname{Cos}^{-1}\left(\frac{11 - 27\,v_i}{16}\right)\right] \qquad (12)$$

with $u_i \in [0, 1]$ when $x \in I_i$ (see also Fig. 3).

Case c. Infinite derivatives at x_i and x_{i+1}

Now $B_i = D_i = 0$ and we are restricted to the cubic parametrization

$$2u^3 - 3u^2 + v_i = 0, \qquad (13)$$

from which we obtain

$$u_i = \tfrac{1}{2} + \cos\left[\frac{4\pi}{3} + \tfrac{1}{3}\operatorname{Cos}^{-1}(1 - 2v_i)\right] \qquad (14)$$

so that $u_i \in [0, 1]$ when $x \in I_i$ (see Fig. 3, Case c).

The 'global' parametrization on $[a, b]$ can now be defined by

$$s = s(u_i(x)) = s(u_i(x)) = x_i + A_i u_i(x) \quad \text{so that} \qquad (15)$$

$$x = x(s) = u_i^{-1}\left(\frac{s - x_i}{A_i}\right). \qquad (16)$$

All the mappings indicated on Fig. 2 have now been provided.

From (15) we see that s simply restores the scale of I_i from $[0, 1]$. Also, $s(x_i) = x_i$ and in fact $s = x$ on $I_i \leftrightarrow$ there are no infinite derivative constraints at x_i and x_{i+1}. Clearly, the first assertion of the theorem is satisfied. Proceeding, we define

$$W(s) = W_i(u_i(s)) \text{ for } s \in I_i. \qquad (17)$$

Since W_i is (and $x(u_i)$ is either linear or) cubic in u_i, which in turn is linear in s, $W(s)$ is (and $x(s)$ is either linear or) cubic in s on I_i, satisfying part 2) of the theorem.

Note that

$$\lim_{s \to x_i^-} \frac{dW}{ds} = \lim_{u_{i-1} \to 1} \frac{dW_{i-1}}{du_{i-1}} \frac{du_{i-1}}{ds} = \frac{C_i}{A_i}$$

$$= \lim_{u_i \to 0} \frac{dW_i}{du_i} \frac{du_i}{ds} = \lim_{s \to x_i^+} \frac{dW}{ds}$$

so that $W(s)$ has continuous first derivatives on (a, b) including the points x_{i_j}, $j = 1, \cdots, k$.

We check for assertion (v) of the theorem. Let $w(x) = W(s(x))$, then

$$w(x_i) = W(s(x_i)) = W(x_i) = w_i \qquad (18)$$

and in fact $w(x) \equiv W(s)$ on $I_i \leftrightarrow$ no infinite derivatives occur at x_i and x_{i+1}.

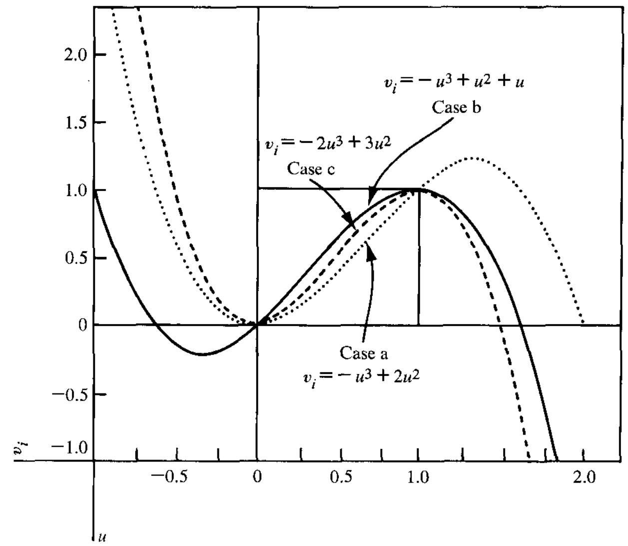

Figure 3 Local parametrization when infinite derivatives occur at the knots.

To examine the differentiability of $w(x)$ at the knots we consider the left,

$$\lim_{x \to x_i^-} \frac{dw}{dx},$$

and right,

$$\lim_{x \to x_i^+} \frac{dw}{dx},$$

drivatives at x_i.

Owing to the possibility different definitions of the local parameter that may prevail on I_{i-1} and I_i, we need to exhaustively consider the distinct circumstances that may arise. This is conveniently done with the aid of the "occurrence matrix" shown in Fig. 4 where, for the sake of clarity, all the different combinations are also pictorialized.

We adopt the convention of referring to the right and left derivative at the endpoints a and b respectively as the derivative there.

Occurrences $1 \rightarrow 3$ were actually incorporated in the construction and for these cases $dw(x_i)/dx = C_i/A_i$. This is also true for 4. Hence at all knots except at x_{i_j} $j = 1, \cdots, k$ $w(x)$ is differentiable and

$$\frac{dw(x_i)}{dx} = \frac{dW(x_i)}{ds} = \frac{C_i}{A_i}.$$

In remaining four occurrences where $x_i = x_{i_j}$ for $j \in \{1, \cdots, k\}$ we find that

$$\lim_{x \to x_{i_j}} \frac{dw}{dx} = \begin{cases} +\infty \text{ if } C_{i_j} > 0 \\ -\infty \text{ if } C_{i_j} < 0 \end{cases}$$

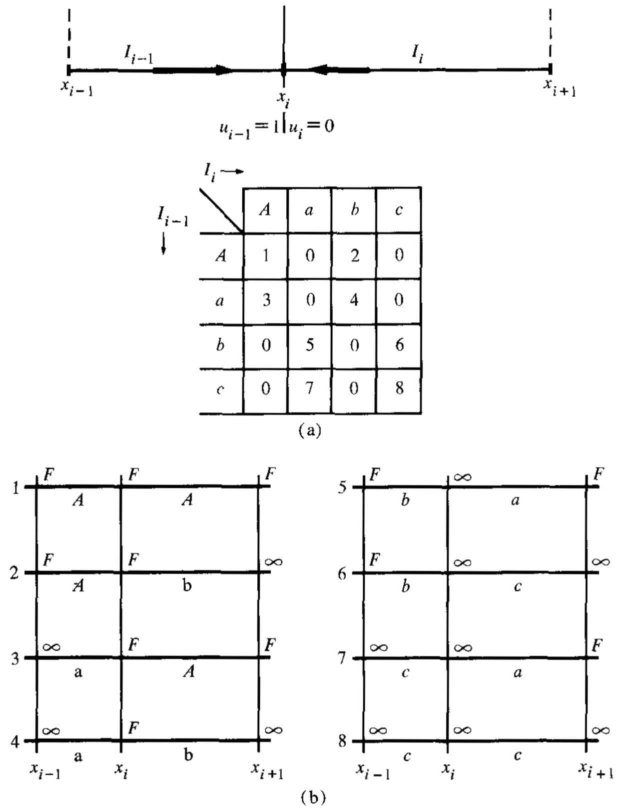

Figure 4 Derivative conditions at endpoints of adjoining intervals. (a) Occurrence matrix for derivative conditions on adjoining intervals. 0 indicates nonoccurrence, 1 → 8 indicates occurrences. (b) Occurrences: F indicates finite derivative, ∞ indicates infinite (±∞) derivative occurring at the knot.

the case $C_i = 0$ having been previously excluded for the existence of an infinite derivative (see discussion after (2)). Subject then to the condition $C_{i_j} \neq 0$, assertion 5) is also satisfied.

- *Determination of $W(s)$*

The C_i are determined so as to render $W(s)$ a spline in the $s - W$ plane. This is equivalent to parts 3) and 4) of the theorem, as well as 2), which has already been shown.

For second order continuity of $W(s)$ at a knot, taken for convenience at x_{i+1}, it is required that

$$\lim_{s\to x_{i+1}^-} \frac{d^2W}{ds^2} = \lim_{s\to x_{i+1}^+} \frac{d^2W}{ds^2},$$

or

$$\frac{1}{A_i^2}\lim_{u_i\to 1} \frac{d^2W}{du_i^2} = \frac{1}{A_{i+1}^2}\lim_{u_{i+1}\to 0} \frac{d^2W_{i+1}}{du_i^2},$$

which yields

$$C_i + H_iC_{i+1} + K_iC_{i+2} = R_i,\ i = 2,\cdots, N-2, \tag{19}$$

where

$$H_i = 2k_i(k_i+1),\ K_i = k_i^2k_{i+1}$$

and

$$R_i = 3[k_i^2(w_{i+2} - w_{i+1}) + (w_{i+1} - w_i)].$$

Two additional conditions are needed for the determination of C_i. Let us consider the integral

$$F(C_1,\cdots, C_N) = \int_a^b \left(\frac{d^2W}{ds^2}\right)^2 ds. \tag{20}$$

When an undeformed thin beam is statically deflected to the shape $W = W(s)$, the strain energy of the deformation (according to linear elasticity theory) is proportional to the integral in (20).

Necessary conditions for the minimization of (20) are that

$$\frac{\partial F}{\partial C_i} = 0,\ \forall i = 1,\cdots, N.$$

Applying the minimization conditions for convenience at $i+1$, we have

$$\frac{\partial}{\partial C_{i+1}} \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} \left(\frac{d^2W}{ds^2}\right)^2 ds$$

$$= \frac{\partial}{\partial C_{i+1}} \sum_{i=1}^{N-1} \frac{1}{A_i^3} \int_0^1 \left(\frac{d^2W_i}{du_i^2}\right)^2 du_i = 0$$

resulting in

$$\int_0^1 \left(\frac{d^2W_i}{du_i^2}\right)\alpha_3''(u_i)\,du_i + k_i^2 \int_0^1 \left(\frac{d^2W_{i+1}}{u_{i+1}^2}\right)\alpha_2''(u_{i+1})\,du_{i+1} = 0,$$

$$i = 2,\cdots, N-2 \tag{21}$$

and

$$\int_0^1 \frac{d^2W_1}{du_1^2}\,\alpha_2''(u_1)\,du_1 = 0,\ \int_0^1 \frac{d^2W_{N-1}}{du_{N-1}^2}\,\alpha_3''(u_{N-1})\,du_{N-1} = 0. \tag{22}$$

Upon evaluation (21) yields (19). From (22) we obtain

$$2C_1 + k_1C_2 = 3(w_2 - w_1),$$
$$C_{N-1} + 2C_N = 3(w_N - w_{N-1}). \tag{23}$$

Defining $C_0 = C_{N+1} = 0$, $H_0 = H_{N-1} = 2$, $k_0 = k_N = 1$, $w_0 = w_1$ and $w_{N+1} = w_N$ enables us to write (19) and (23) in the compact form

$$C_i + H_iC_{i+1} + K_iC_{i+2} = R_i\ i = 0,\cdots, N-1. \tag{24}$$

Interestingly enough (23) also imply that

$$W_+''(a) = W_-''(b) = 0.$$

The C_i, and hence $W(s)$, are uniquely determined by (24) since the coefficient matrix Δ of (24) is nonsingular [5]. In fact $\Delta^{-1} = [a_{st}]$, where

$$a_{st} = (-1)^{s+t} \sum_{r=\max(s,\,t)}^{N-1} \left(\prod_{n=s}^{r} K'_n\right) \Big/ \left(K'_r \prod_{n=t}^{r} H'_n\right)$$

with

$$H_1' = H_1,\ K'_{i-1} = K_{i-1}/H'_{i-1},\ H'_{i-1} = H_i - K'_{i-1},$$
$$i = 2, 3, \cdots, N-1.$$

It turns out that $a_{st} \neq 0\ \forall s, t = 0, 1, \cdots, N-1$.

We appeal to the celebrated result (quoted in [1], p. 3, p. 75):

Theorem (*Holladay*) Of all functions having continuous second derivatives on (a, b) and passing through all the knots, the spline $W(s)$ having $W''_+(a) = W''_-(b) = 0$ minimizes (20).

Hence, (24) provides necessary and sufficient conditions for the minimization of the integral.

Assertions 1) $\rightarrow$ 4) of the theorem as well as a part of 5) have been proved. We need conditions to guarantee that $C_{i_j} \neq 0$, $\forall_j = 1, \cdots, k$. These are given by (*), obtained directly from (24) via Cramer's rule. q.e.d.

Implementation

The function $w(x)$ is not a spline. It just misses being a spline on the intervals I'_i. There it is obtained "homeomorphically"—in the sense of $w(x) = W[s(x)]$—from a spline. Further, $w''(x)$ exists and is continuous on $[a, b]'$. The reader interested in the determination of cubic splines when other constraints not involving infinite derivatives are imposed is referred to the cited work of Dimsdale.

The salient feature of the algorithm is the homeomorphism s. The spline $W(s)$ in the $s - W$ plane is identical to the one constructed in the $x - w$ plane by Dimsdale's algorithm when only finite derivatives exist at the knots. This is advantageous, because if additional constraints on the spline are required, these can be incorporated in $W(s)$ directly via Dimsdale's method. Since $W(s) = w(x)$ on $I_i \neq I'_i$, the constraints carry over to the $x - w$ plane. In addition, we obtain $w(x)$ in closed form throughout the interval $[a, b]$ including the subintervals where infinite derivatives exist.

We elaborate now on the partition invariant property of $w(x)$ alluded to in the introduction. Suppose that $w(x)$ is constructed on $[a, b]$ with a partition $P[a, b] = \{a = x_1 < x_2 < \cdots < x_N = b\}$—we write w_P for emphasis—and it is desired to alter w_P on I_i. This situation arises very frequently in the applications where a designer feels he has achieved the desired shape on $[a, b]$ except for I_i. He wishes to modify w_P on a portion of I_i without altering it elsewhere. The way this is done is by introducing additional knots at y_m, with $m = 1, \cdots, M$ and $x_i < y_1 < y_2 < \cdots < y_M < x_{i+1}$, so that the resulting spline resembles more closely the desired shape. Physically, this imitates the bending of a thin beam, according to the

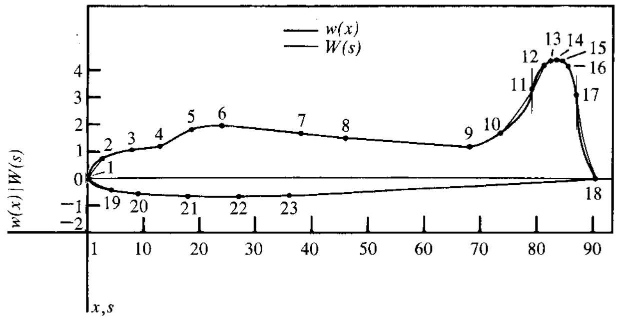

Figure 5 Airplane silhouette constructed in four parts (see also Table 2). Vertical slopes at knots number 1, 11 and 17. Slope discontinuities at knots 4 and 9 are intentional. Note differences on intervals with vertical slopes between $w(x)$ and $W(s)$.

Table 2 Values for curve in Fig. 5.

	Point Number	x_i	w_i	C_i	∞-Derivative
Spline 1	1	0	0.0	0.798	Yes
(nose)	2	2.6	0.7	1.009	
	3	7.8	1.0	0.055	
	4	13	1.2		
Spline 2	4	13	1.2	0.720	
(body)	5	18.5	1.8	0.360	
	6	24.0	1.9	−0.146	
	7	37.8	1.6	−0.150	
	8	46.0	1.5	−0.244	
	9	68.0	1.1		
Spline 3	9	68.0	1.1	0.264	
(tail)	10	73.5	1.6	0.971	
	11	79.0	3.2	0.861	Yes
	12	81.2	4.0	0.302	
	13	82.3	4.2	0.080	
	14	83.4	4.2	−0.022	
	15	84.5	4.2	0.009	
	16	85.6	4.0	−0.614	
	17	86.7	3.0	−3.464	Yes
	18	90.0	0.0		
Spline 4	1	0	0.0	0.476	Yes
(belly)	19	4.5	0.4	0.248	
	20	9.0	0.5	0.065	
	21	18.0	0.58	0.058	
	22	27.0	0.6	0.004	
	23	36.0	0.6	−0.077	
	18	90.0	0.0		

"small" deflections theory, and is equivalent to forcing the beam to pass through the additional knots by placing simple supports there. On $J_q = [y_q, y_{q+1}] \subset I_i$ the shape of w_P is to be preserved, and this may be the case for any number of subintervals. We have then a new partition $\overline{P}[a, b] = P[a, b] \cup \{y_1, \cdots, y_M\}$ and we wish to construct $w_{\overline{P}}$ so that $w_{\overline{P}}(x) = w_P(x)$ $\forall x \in Q = [a, x_i] \cup J_q \cup [x_{i+1}, b]$.

If only finite derivatives occur at the knots the construction is easy. On $[a, x_i] \cup [x_{i+1}, b]$ we retain the cubic portions $W_n(s(x))$ of $w_P(x)$, $n = 1, \cdots, I_{i-1}, I_{i+1}, \cdots, N$ and let $w_{\overline{P}}(x) = W_n(s(x))$ there. On J_q we specify $W_q(s(x))$ as the cubic passing through $(y_q, w_P(y_q))$ with slope $w'_P(y_q)$ and through $(y_{q+1}, w_P(y_{q+1}))$ with slope $w'_P(y_{q+1})$. Clearly, $W_q(s(x)) = W_i(s(x))$ on J_q since a cubic is uniquely determined by specifying two points it must pass through with given derivatives there. We let $w_{\overline{P}}(x) = W_q(s(x))$ on J_q. Finally we specify that

$$\begin{cases} w^L_{\overline{P}}(x_i) = w'_P(x_i),\ w^L_{\overline{P}}(y_q) = w'_P(y_q) \\ w^L_{\overline{P}}(y_{q+1}) = w'_P(y_{q+1}),\ w^L_{\overline{P}}(x_{i+1}) = w'_P(x_{i+1}). \end{cases} \quad (25)$$

Then using Dimsdale's method $w_{\overline{P}}(x)$ is constructed on $[x_i, y_{q-1}] \cup [y_{q+1}, x_{i+1}]$ subject to the constraints in (25). Hence $w_{\overline{P}}(x) = w_P(x)$ on Q.

When infinite derivatives occur, however, this process fails since on I_i', $W_i(s(x))$ is not a cubic in x. However, here the $s - W$ plane comes to the rescue. For $W_P(s)$ is always piecewise cubic and has finite derivatives everywhere on $[a, b]$. So by the process we have just outlined $W_{\overline{P}}(s)$ is constructed with $W_P(s) = W_{\overline{P}}(s)$ on Q. Then using the u_i derived for partition P (*not* $\overline{P}$) we obtain $w_{\overline{P}}$ with $w_{\overline{P}}(x) = w_P(x)$ on Q. That is w_P is "invariant" on a specified $Q \subset [a, b]$ with respect to changes in the partition $P[a, b]$.

This property was crucially important in the application [3] that motivated this work. There some subintervals, where a $w(x)$ was constructed, needed to be "shrunk" sufficiently to guarantee convergence of a certain series expansion, this without changing the shape of $w(x)$.

The algorithm given was tried on several examples. One, constructed for convenience in four separate parts, is shown in Fig. 5 and Table 2. It was found that the algorithm is computationally efficient and easy to implement. Condition (*) poses no problems in implementation. Rather it implies that the knots ought to be chosen judiciously.

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References

1. J. H. Ahlberg, E. N. Nilson and J. L. Walsh, *The Theory of Splines and their Applications*, Academic Press, New York 1967.
2. B. Dimsdale and K. Johnson, "Multiconic Surfaces, *IBM J. Res. Develop.* **19** 523 (1975).
3. A. Inselberg, "Variable Geometry Cochlear Model at Low Frequency Input," submitted to *IBM J. Res. Develop.*
4. B. Dimsdale, "On Multiconic Surfaces," *Report No. G320.2661*, IBM Scientific Center, Los Angeles, California 1974.
5. B. Dimsdale, "On Spline End Condition Calculations," *Report No. G320.2656*, IBM Scientific Center, Los Angeles, California 1972.
6. G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, Revised Edition, Macmillan, New York 1953, p. 96.

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