Natural quadrics: Projections and intersections

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Geometrical modelers usually strive to support at least solids bounded by the results of Boolean operations on planes, spheres, cylinders, and cones, that is, the natural quadrics. Most often this set is treated as a subset of the set of quadric surfaces. Although the intersection of two quadrics is a mathematically tractable problem, in implementation it leads to complexity and stability problems. Even in the restriction to the natural quadrics these problems can persist. This paper presents a method which, by using the projections of natural quadrics onto planes and spheres, reduces the intersection of two natural quadrics to the calculation of the intersections of lines and circles on planes and spheres. In order to make the claims of the method easily verifiable and provide the tools necessary for implementation, explicit descriptions of the projections are also included.

Introduction

The solid modelers used in mechanical part design and analysis most often support objects bounded by regions of surfaces drawn from a very simple class of surfaces: the natural quadrics—planes, spheres, (right circular) cylinders, and (right circular) cones [1]. If a modeler is to make explicit use of the boundary of an object it processes, it must be able to compute this boundary; at a minimum, this implies the ability to compute the intersection of the surfaces involved in the definition of the object. The intersection of surfaces in singular positions is an unstable problem, as can easily be seen by considering two cylinders sharing a common line. Moving the two slightly apart yields two cylinders with no intersection; if they are moved slightly closer together, the one line of intersection becomes two. If one cylinder is rotated slightly in one direction, the intersection is reduced to a point, but if rotation occurs in a different direction, the intersection can become an arbitrarily large closed curve. In sum, the slightest perturbation of either cylinder can cause their intersection to undergo profound change. Given the potential consequences of this inherent instability and the simplicity of the surfaces involved, the search for some closed-form analytic solution is appealing. Yet, their simplicity notwithstanding, it has proved to be a surprisingly difficult task to find a computationally tractable method to obtain these intersections exactly.

This paper addresses the problem of finding an exact closed-form solution for the intersection of two natural quadrics and solves it in what we believe is a conceptually and computationally simple manner. We begin in Section 1 by giving a cursory discussion of the two main theoretical methods applicable to the problem. These methods in fact solve a natural generalization of the problem, determining the intersection of two quadric surfaces, that is, surfaces defined by second-degree polynomial equations in three variables. Although they are elegant mathematical solutions, in their full generality they pose problems for use in geometrical modeling which are discussed here. When the methods are specialized to natural quadrics, one of them, due to Levin [2], simplifies greatly, and we study it in detail. Except in the trivial intersections involving only planes and spheres, it becomes a method of intersecting a cylinder or cone with an arbitrary natural quadric that proceeds by viewing the former as a ruled surface, a surface generated by a family of lines through a base curve. This reduces the calculation of the intersection of the two natural quadrics to a one-parameter family of line and quadric surface intersections. A completely satisfactory representation of the intersection results if the domain of this one-parameter family, that is, the domain of the base curve, can be partitioned into segments that define smooth curves in the intersection. Computation of such a partition, called a domain segmentation, can be achieved in a straightforward manner; in general, however, it requires the solution of a fourth-degree polynomial. In practice, this equation is only approximated, or it is avoided by a case-by-case analysis, or some combination of these two approaches is used. We present a different approach based on solving a related but simpler problem: Rather than producing a domain segmentation directly, we partition the base curve itself in a similar manner, yielding a base curve segmentation. We show that a base curve segmentation can be found by intersecting a circle that serves as the base curve of the cylinder or cone with a small number of circles, lines, and points that define an appropriately partitioned projection of the second natural quadric onto a plane or a sphere containing the base curve. The first section closes by showing that a base curve segmentation is sufficient by proving that it leads directly to a domain segmentation. At this point we have shown that by completing Levin's method with the calculation of a domain segmentation using projections, the intersection of two natural quadrics can be obtained exactly and explicitly by performing computations never exceeding the difficulty of computing the intersection of two circles.

The technique presented in Section 1 presupposes access to explicit descriptions of the projections of arbitrary natural quadrics onto arbitrary planes and spheres. It further presupposes that this projection has been partitioned into regions such that any two points in the region have pre-images with the same cardinality. Moreover, the regions and the projections themselves must be described in terms of a small number of circles, lines, and points. After establishing some notation in Section 2, we turn to a presentation of just such a description. In Section 3 we treat the planar projections, and in Section 4 the spherical projections. Given the long history of the study of quadric surfaces, it is likely that these projections have been considered before, especially since many of them are quite trivial to obtain. However, since there seems to be no ready source which discusses them completely and in the detail that is required in this study, and since their explicit form is necessary for our arguments, we have presented them in their entirety with proofs of their derivation.

In Section 5 the intersection of an arbitrary cone and an arbitrary cylinder is considered by an exhaustive case-by-case analysis. In a real sense the results of Section 1 make this type of analysis superfluous. By appeal to projections and lower-dimensional modeling utilities that can intersect circles with planar and spherical regions bounded by circles, lines, and spheres, this problem can more easily be solved. Yet, while recognizing that an example is not a proof, we believe that this example partially demonstrates some further potential uses of the technique. In this section we assume that the computations are to be performed in a system allowing high-precision rational calculation, but do not assume the ability to reliably manipulate or evaluate the signs of irrational quantities. The projections themselves in general involve a radical, even if we begin with surfaces described by rational data. If we duplicate the calculations of a modeling system by intersecting the base curve with the curves describing the projection, more irrationalities arise. Instead, we develop polynomials in the parameters of the problems which we use to partition the space of parameters, itself, into regions where the intersection can be easily known and explicitly given. This has several advantages. First, the original intersection problem can now be trivially solved by evaluating the sign of at most six polynomials, after which the intersection can be produced directly. Second, because we have partitioned by the signs of polynomials and considered the consequences of all possible sign combinations, we can be assured of avoiding the bane of all case-by-case analyses: missing a case. Third, if a quadric is to be approximated by another, it seems reasonable to require that the approximation not change the topology of the intersections involving the quadric. The signs of the polynomials give a ready means to test for this. Alternatively, we could try to search in the region for a new set of parameter values for an approximation. Finally, we see that within any subset of the partition the intersection problem becomes stable in contrast to the general situation.

The paper closes by summarizing the results and identifying certain simple generalizations.

1. Intersection of natural quadrics

Two main theoretical tools are applicable to the problem of computing the intersection of two natural quadrics. Each addresses a natural generalization: computing the intersection of two arbitrary quadric surfaces. The first is based on the work of Levin [2] in rendering the intersection of two quadric surfaces, and the second on that of Ocken, Schwartz, and Sharir [3]. The Levin method proceeds by finding a ruled quadric surface in the pencil of the two quadric surfaces which is used to calculate the intersection as a one-parameter family of intersections of a line and a quadric surface. The auxiliary ruled quadric surface is identified by computing the roots of a determinantal equation, which yields a third-degree polynomial equation, and evaluating the determinants of several matrices of second through fourth order dependent on the roots and the original surfaces. The method of Ocken, Schwartz, and Sharir begins by obtaining the eigenvalues and generalized eigenvectors of a four-by-four (generally) nonsymmetric matrix, dependent on the parameters of the problem, and then uses them to obtain a decomposition of the four-dimensional space on which the matrix acts. The decomposition allows the intersection to be represented as one of a small number of intersection problems.

Although both of these general methods are mathematically elegant solutions, they suffer from similar limitations for use in a geometrical modeler. First, there is the problem of complexity. The intersection of surfaces is a highly unstable problem, so that any error in calculation can cause profound changes and logical inconsistencies. As should be expected, both methods mirror this instability, so that any approximation scheme would be problematical for either method. On the other hand, using the exact calculations needed in either case as the core routines of a geometrical modeler seems to be beyond the power of the current generation of symbolic manipulators. Second, to be useful in a modeler the intersection should be segmented into a small number of disjoint curves and points whose topological, geometric, and relational structures are known or can be easily ascertained. While each method is in a sense complete, neither yields the intersection in a form that is complete enough for a modeler. The one-parameter family of intersections between lines and a quadric that is the result of Levin's method has the domain of the base curve of the auxiliary ruled quadric as its parameter domain. The intersection of a line and a quadric surface can be zero, one, or two points or the line itself, so that each parameter value may correspond to zero, one, or two points or a line of intersection. The partition of the parameter domain into maximal connected subsets with each subset corresponding to one of these four intersection types can be used to segment the intersection into useful parts, but this can generally require the solution of high-degree polynomial equations. Because the second method uses projective transformations in three-space, it treats the intersection problem by solving a projectively equivalent problem, yielding the final result as a collection of points and curves which is a subset of a sphere, a one-sheeted hyperboloid of revolution, or a cylinder, and which is projectively equivalent to the original intersection set. A projective equivalence may map disjoint curves to intersecting curves, planar curves to infinity, or bounded curves to unbounded ones. Obtaining a complete description of the true affine intersection could require calculations involving the projectively equivalent curves, the projective transformations, and the surfaces themselves. Thus each method would require a different type of nontrivial postprocessing, possibly by careful approximation methods, but most likely by further symbolic calculation.

In sum, their complexity and need for further nontrivial processing make the general methods unattractive to implementers of geometrical modelers.

Even in the special case of natural quadrics one finds little simplification in the second method; however, Levin's method is greatly simplified in this case. Since the intersection of two natural quadrics is either the intersection of two spheres (which can easily be solved) or the intersection must already involve a ruled surface, there is no need to search for an auxiliary ruled quadric. Thus one may avoid the computation of roots and evaluations of determinants usually associated with this method, and Levin's method reduces to a technique for intersecting a ruled natural quadric with an arbitrary natural quadric. Because of this simplicity for natural quadrics, either explicitly [4] or implicitly, some variant of Levin's method is most commonly used in practice, and yet a serious problem still exists. To follow Levin's technique further and to explain this problem, we need more explicit information about representing cones and cylinders as ruled surfaces.

A ruled surface is represented by choosing a parameterized base curve on it, $\mathrm{b}: t \in \mathrm{domain(b)} \rightarrow \mathrm{b}(t) \in \mathbb{R}^3$, and describing the surface as the union of the parameterized lines, $\mathrm{L}(s;t)$, through the base curve. If T is a cylinder or cone, then b is generally and most simply chosen to be some parameterized circle on T. To be more concrete, b is parameterized in terms of the transcendentals, $\sin(t)$ and $\cos(t)$, or in terms of the equivalent rationals, $(1-t^2)/(1+t^2)$ and $2t/(1+t^2)$, so that b is parameterized as $\mathrm{b}: t \in [0, 2\pi) \rightarrow \mathrm{p} + r[\cos(t)\mathrm{u} + \sin(t)\mathrm{u}']$ or $\mathrm{b}: t \in \mathbb{R} \rightarrow \mathrm{p} + r(1+t^2)^{-1}[2t\mathrm{u} + (1-t^2)\mathrm{u}']$, where p is a point on the axis of T (not equal to the vertex of T, if it is a cone), and u and u′ are unit vectors such that u, u′, and the axis of T are mutually orthogonal [5]. Thus, if T is a cylinder and v is a vector parallel to its axis, then each line in T is represented as $\mathrm{L}(s;t) = \{\mathrm{b}(t) + s\mathrm{v}: s \in \mathbb{R}\}$, for some $t \in \mathrm{domain(b)}$, while if T is a cone with vertex q, then each line in T is represented as $\mathrm{L}(s;t) = \{\mathrm{q} + s\mathrm{b}(t): s \in \mathbb{R}\}$, and in either case T itself is represented [6] as $\mathrm{T} = \{\mathrm{L}(s;t): t \in \mathrm{domain(b)}, s \in \mathbb{R}\}$.

Now let S and T be natural quadrics. $\mathrm{S} \cap \mathrm{T}$ is easily obtained if both S and T are a plane or a sphere, and so we ignore these simple cases by assuming T to be a cylinder or cone. Since S is a quadric surface, S can be represented as the solution set of a second-degree polynomial in three variables, $f: \mathrm{x} \in \mathbb{R}^3 \rightarrow {}^t\mathrm{xQx} + {}^t\mathrm{yx} + c$, where Q is a symmetric matrix, y is a vector, and c is a scalar, which all depend on S. Now for any t in domain(b), $\mathrm{L}(s;t) \cap \mathrm{S}$ is found by solving for s in $f(\mathrm{L}(s;t)) = A(t)s^2 + B(t)s + C(t) = 0$, where $A(t) = {}^t\mathrm{vQv}$, $B(t) = 2\,{}^t\mathrm{vQb}(t) + {}^t\mathrm{yv}$, and $C(t) = {}^t(\mathrm{b}(t))\mathrm{Qb}(t) + {}^t\mathrm{yb}(t) + c$, if T is a cylinder, while $A(t) = {}^t(\mathrm{b}(t))\mathrm{Qb}(t)$, $B(t) = 2\,{}^t\mathrm{qQb}(t) + {}^t\mathrm{yb}(t)$, and $C(t) = {}^t\mathrm{qQq} + {}^t\mathrm{yq} + c$, if T is a cone. If $f(\mathrm{L}(s;t))$ degenerates to the zero function, then every s solves it, so that $\mathrm{L}(s;t) \cap \mathrm{S} = \Gamma_1(t) = \mathrm{L}(s;t)$. If only $A(t)$ and $B(t)$ degenerate to zero, then no s solves the equation, and the intersection is empty. If $A(t) = 0$, but $B(t) \neq 0$, then

$s = -C(t)/B(t)$ is the sole solution, and $\Gamma_2(t) = \mathrm{L}(-C(t)/B(t); t)$ is the sole point of intersection. Finally, if $A(t) \neq 0$, and we denote the discriminant, $(B(t))^2 - 4A(t)C(t)$, by $\delta(t)$, then, of course, there are no points of intersection, if $\delta(t) < 0$; $\Gamma_3(t) = \mathrm{L}(-B(t)/(2A(t)); t)$ is the sole point of intersection, if $\delta(t) = 0$; and $\Gamma_4^{\pm}(t) = \mathrm{L}((-B(t) \pm (\delta(t))^{1/2})/(2A(t)); t)$ are the two points of intersection, if $\delta(t) > 0$. If $\mathrm{L}(s; t) \cap \mathrm{S}$ is empty, then we will say that t has parameter value type of zero. Similarly, if $\mathrm{L}(s; t) \cap \mathrm{S} = \Gamma_i(t)$ for i = 1, 2, or 3, then we will say that t has parameter value type of i, and we define t to have parameter value type four, if $\mathrm{L}(s; t) \cap \mathrm{S} = \Gamma_4^{\pm}(t)$. Thus, for a fixed $t \in$ domain(b), by testing whether $A(t)$, $B(t)$, and $C(t)$ are zero, and possibly computing the sign (−, 0, or +) of $\delta(t)$, we can find its parameter value type and so have an explicit formula for $\mathrm{L}(s; t) \cap \mathrm{S}$.

Since the sign of the functions $A(t)$, $B(t)$, and $C(t)$ can change over domain(b), the parameter value type or even the topological type of $\mathrm{L}(s; t) \cap \mathrm{S}$ may change over domain(b), so that we cannot expect to find a general formula expressing $\mathrm{L}(s; t) \cap \mathrm{S}$ as a function of t valid over all of domain(b). However, if g is a connected subset of domain(b) on which the parameter value type does not change, then much more follows. First assume that T is not a cone with its vertex on S, so that no two lines in the ruling of T share a common point of intersection with S. If the parameter value type of all t in g is zero, then the lines through g are disjoint from S. If the parameter value type of all t in g is one and g has a nonempty interior, then S and T coincide over a two-dimensional subset, and hence S = T. If the parameter value type of all t in g is two (or three), then $t \in \mathrm{g} \rightarrow \Gamma_2(t)$ [or $\Gamma_3(t)$] defines a curve in $\mathrm{S} \cap \mathrm{T}$ that is smooth on the interior of g, since $A(t)$, $B(t)$, and $C(t)$ are smooth functions of t. Finally, if the parameter value type of all t in g is four, then $t \in \mathrm{g} \rightarrow \Gamma_4^{\pm}(t)$ define two disjoint curves in $\mathrm{S} \cap \mathrm{T}$ that are smooth on the interior of g. Now let T be a cone with its vertex, q, on S. Since q is always in $\mathrm{L}(s; t) \cap \mathrm{S}$, the parameter value type of t in g cannot be zero. If the parameter value type of t in g is one, then the interpretation is the same as above. If the parameter value type of t in g is two or three, then, since q belongs to the intersection of each line with S, the associated parameterized curves, Γ_2 and Γ_3, must degenerate to the constant map, $t \in \mathrm{g} \rightarrow \mathrm{q}$. Since q belongs to S, q must satisfy the defining equation of S, so that $C(t)$ must be identically zero, which implies that $(-B(t) \pm (\delta(t))^{1/2})/(2A(t))$ reduces to zero and $-B(t)/A(t)$. Since by the choice of parameterization zero corresponds to q, we find that if all $t \in \mathrm{g}$ have parameter value type four, then the contribution to $\mathrm{S} \cap \mathrm{T}$ of the lines through g is q and the curve, $t \in \mathrm{g} \rightarrow \Gamma_5(t) = \mathrm{L}(-B(t)/A(t); t)$, which is smooth on the interior of g and disjoint from q. Thus in any case if g is a connected subset of domain(b) on which the parameter value type does not change, then we can immediately find the part of $\mathrm{S} \cap \mathrm{T}$ corresponding to g. This result in its simplicity, generality, and conciseness is very appealing, and so leads naturally to a search for a partition of domain(b) into finitely many connected subsets such that on each of these subsets the parameter value type is constant. We refer to such a partition as a domain segmentation. If we can find an exact and explicit domain segmentation, then the above implies that we can find an exact and explicit representation of $\mathrm{S} \cap \mathrm{T}$. Thus, if we can produce an exact and explicit domain segmentation in a computationally tractable manner, then we can say that we have exactly and explicitly solved the problem of intersecting two natural quadrics.

Clearly, the obvious method of producing a domain segmentation is to compute the roots of $A(t)$, $B(t)$, $C(t)$, and $\delta(t)$, and use them appropriately to define a partition of domain(b), but herein lies the problem. If the rational parameterization is employed, then in general $B(t)$ is a rational quadric in t, and either $A(t)$ or $C(t)$ is a rational quartic in t, with the other being constant in t, so that $\delta(t)$ is also a rational quartic; thus, finding the roots of $A(t)$ or $C(t)$ and $\delta(t)$ is equivalent to solving fourth-degree polynomial equations. Using the transcendental parameterization is no better. This leads to quadratic equations in $\sin(t)$ and $\cos(t)$, whose solutions again lead to fourth-degree polynomial equations, but this time in $\sin(t)$ [or $\cos(t)$], which adds the burden of the evaluation of an inverse trigonometric function. Closed-form solutions of fourth-degree equations exist, but the complexity is daunting, and in practice the roots can be and are only approximated [4], which can cause large errors in singular situations. Thus, employing this straightforward approach to solve the key problem of finding an explicit domain segmentation seems to yield a problem which is not computationally tractable in the general setting.

Others have recognized the problems in the application of the theoretical methods of quadric and natural quadric intersection and have attacked them and related problems by many methods. Morgan [7] has suggested exploiting more general techniques for approximating solutions to systems of polynomial equations that are based on homotopy continuation methods. Farouki et al. [8] have mixed classical algebraic geometry with modern polynomial factorization to explore recognition of simpler singular intersections and their parameterization. Sarraga [4], using a formulation equivalent to the one presented here, has applied analytic and algebraic geometric techniques to simplify the equations in special cases and then to solve them explicitly or by standard approximation packages. Miller [9] has written an exhaustive case-by-case analysis based on geometric invariants for nonplanar intersection curves.

We offer a conceptually and computationally simple alternative based on partitioning the base curve itself, rather than its domain, by using projections, which completely and explicitly solves the domain segmentation problem using only trivial geometric calculations. As before, let S and T be natural quadrics, with T being a cylinder or a cone, and let b be a parameterization of a circle on T. Let $\hat{b}$ be the circle, itself, that is the image of b. Let p be a point of $\hat{b}$. If the line in the ruling of T through p is disjoint from S, define p to be of type 0; if the line belongs to S, define p to be of type 1; if it intersects once, define p to be of type 2–3; and if it intersects twice, define p to be of type 4. Define a base curve segmentation to be a partition of $\hat{b}$ into finitely many connected subsets each of which is either a point or an open arc of constant type points. Clearly, a domain segmentation immediately yields a base curve segmentation, but as we will see, it is much easier to reverse the process.

In Section 3 we present explicit formulas for the perpendicular projection of an arbitrary natural quadric onto an arbitrary plane, and moreover note the cardinality of the pre-image of each point in the projection. In every case the projection can be described by a set of at most three circles, lines, or points in the plane, and this same set suffices to partition the projection into regions in which all the points have pre-images with the same cardinality, which we refer to as a partitioned projection. If T is a cylinder, let P be the plane containing $\hat{b}$, and let $\Pi'_P(S)$ be a partitioned projection of S onto P. Since a line in T intersects S in n points, if and only if the line intersects $\Pi'_P(S)$ in a point whose pre-image has cardinality k, a base curve segmentation can now be produced by intersecting $\hat{b}$ with the circles, lines, and points that define $\Pi'_P(S)$ and using the points of intersection to partition $\hat{b}$. These intersections can be performed in terms of some parameterization of $\hat{b}$ or directly by simple geometric arguments.

In Section 4 we present explicit formulas for the spherical projection of an arbitrary natural quadric onto an arbitrary sphere, and again note the cardinality of the pre-image of each point in the projection. In each case the projection can be described by a set of at most five circles or points on the sphere, and this same set suffices to determine a partitioned projection. If T is a cone, let S' be the sphere centered at q, the vertex of T, and containing $\hat{b}$, and let $\Pi'_{S'}(S)$ be a partitioned projection of S onto S'. Let $\hat{b}_-$ be the circle antipodal to $\hat{b}$, that is, $\hat{b}_- = \{2q - x : x \in \hat{b}\}$. A base curve segmentation can now be produced by intersecting $\hat{b}$ and $\hat{b}_-$ with the circles and points that define $\Pi'_{S'}(S)$ and observing whether $q \in S$. In particular, let $\mathcal{P}$ be the union of the isolated points of intersection that $\hat{b}$ has with the points and circles that define $\Pi'_{S'}(S)$. Let $\mathcal{P}_-$ be the union of the isolated points of intersection that $\hat{b}_-$ has with the points and circles that define $\Pi'_{S'}(S)$, and let $\mathcal{P}^-$ be the set of points antipodal to points of $\mathcal{P}_-$; that is, $x \in \mathcal{P}^-$, if $2q - x \in \mathcal{P}_-$. Now use $\mathcal{P} \cup \mathcal{P}^-$ to partition $\hat{b}$ into points and open arcs. If γ is one of the open arcs of this partition, then γ is a subset of one region in $\Pi'_{S'}(S)$, and the cardinality of the pre-images of points in this region is k, if and only if each open half-line of T emanating from q and passing through γ intersects S in k points. Moreover, if $\gamma_- = \{2q - x : x \in \gamma\}$, then γ_- is a subset of one region in $\Pi'_{S'}(S)$, and the cardinality of the pre-images of points in this region is k_-, if and only if each open half-line of T emanating from q and passing through γ_- intersects S in k_- points. Thus, each line in T passing through γ intersects S in $k + k_- + 1$ points, if $q \in S$, and in $k + k_-$ points, otherwise. The required intersections can be performed in terms of parameterizations of $\hat{b}$ and $\hat{b}_-$ or directly by simple geometric arguments. For example, if $\mathcal{C}$ is one of the circles that define $\Pi'_{S'}(S)$, then let $P_{\mathcal{C}}$ be the plane containing it, and let $P_{\hat{b}}$ be the plane containing $\hat{b}$. Now $\mathcal{C} \cap \hat{b} = P_{\mathcal{C}} \cap P_{\hat{b}} \cap S'$, which is nothing more than a line-sphere intersection.

At this point we have shown that by using projections we can always produce a base curve segmentation by performing a small number of trivial intersection problems, never more complex than a circle–circle intersection. Now we show that a base curve segmentation leads directly to a domain segmentation. For this purpose assume that a parameterization, b, has been chosen for $\hat{b}$, and that we have computed a base curve segmentation of $\hat{b}$. If we have used the parameterization to find the base curve segmentation, then we also have a partition of domain(b); if not, then either applications of the arccos and arcsin for the transcendental parameterization or solutions of the associated second-degree polynomial equations for the rational parameterization yield a partition of domain(b). In fact, this partition is a domain segmentation, as we now show. Let γ be one of the connected components in the base curve segmentation, and let g be the associated connected subset of the partition of domain(b). If T is a cone with its vertex on S, then the point type of points of γ determines a unique parameter value type for the values in g, so that the partition is a segmentation. Thus, assume that T is not a cone with its vertex on S. It is obvious that the points in γ are of type 0, 1, or 4, if and only if the parameter values of g have parameter value type 0, 1, or 4, respectively, so assume that the points in γ are of type 2–3. Clearly, each point of γ must correspond to a parameter value of type 2 or 3. If γ is a single point, then g is a single value, and $\gamma = b(g)$, so that g is of parameter value type 2, if $A(g) = 0$, and is of type 3, otherwise. Now assume that γ is an open arc, and that t_0 has parameter value type 3, for some $b(t_0) = p \in \gamma$, so that $A(t_0) \neq 0$ and $\delta(t_0) = 0$. Since A is continuous, it must be nonzero in some neighborhood of t_0. If $\delta(t)$ is not identically zero in this neighborhood, then any subneighborhood of t_0 must contain parameter values of type 0 or 4, contradicting the assumption that γ contains only points of type 2–3. Thus δ must be identically zero in a neighborhood, and hence everywhere, and in particular, in g. If there exists $t' \in g$ such that $A(t') = 0$, then since $\delta(t') = 0$, $B(t') = 0$, but this implies that t' has parameter value type 0 or 1, again contradicting the assumptions on γ. Thus, for all $t \in g$ we have that $A(t) \neq 0$ and $\delta(t) = 0$, so that each $t \in g$ has parameter value type 3. Since we can distinguish between parameter value types 2 and 3 by testing A, we find that if γ is a type 2–3 open arc, then all elements of g have parameter value type 2, if $A(t_0) = 0$ for any $b(t_0) \in g$; otherwise, they have parameter value type 3. In conclusion, we see that the partition of domain(b) determined by a base curve segmentation is indeed a domain segmentation, as claimed.

To summarize, we began this section with a cursory look at the two major theoretical methods for quadric surface intersection. One of these, Levin's method, was seen to simplify in the special case of natural quadrics, reducing to a technique for intersecting a ruled natural quadric with an arbitrary one. We examined this method in detail, and showed that an intersection can be described explicitly in terms of finitely many disjoint smooth parameterized curves and points, if a domain segmentation is found. We then showed that a related concept, a base curve segmentation, can always be found by exploiting an appropriate partitioned projection to segment a base curve. Since these projections are describable by small numbers of circles, lines, and points, it followed that this base curve segmentation could be produced by calculations as simple as those required in circle–circle intersection. Finally, we showed that a base curve segmentation leads directly to a domain segmentation; hence, by utilizing these techniques we always can explicitly and exactly find the intersection of two natural quadrics by means of tractable geometric calculations. In fact, if we assume the existence of modeling utilities in a system that supports objects in planes and spheres bounded by circles, lines, and points that can intersect such objects with circles, and assume the ability to evaluate polynomial functions, then we can use these to determine completely the exact and explicit intersection of two natural quadrics.

2. Notation

• *General notation*

All vectors are written as lowercase roman characters, and all scalars are written as lowercase italic characters; for example, p is a vector and r is a scalar.

If A is a point set, $\bar{A}$ denotes its topological closure.

Let f be a function with domain X and range Y. If Z is a subset of Y, then f is referred to as an n-fold map onto Z, if n is the cardinality of the pre-image of each $z \in Z$.

• *Linear notation*

If w is a nonzero vector, let L(w, p) denote the affine line, $\{p + t w : t \in \mathbb{R}\}$, $L^+(w, p)$ the open half-line, $\{p + t w : t > 0\}$, and $L^-(w, p)$ the open half-line, $\{p + t w : t < 0\}$. Let P(w, p) denote the affine plane, $\{x \in \mathbb{R}^3 : {}^t w(x - p) = 0\}$, and $P^+(w, p)$ and $P^-(w, p)$ denote the related half-spaces $P^+(w, p) = \{x \in \mathbb{R}^3 : {}^t w(x - p) > 0\}$, and $P^-(w, p) = \{x \in \mathbb{R}^3 : {}^t w(x - p) < 0\}$.

Let P = P(w, p) be a plane. For $r > 0$, if $p' \in P$, let $\mathcal{C}(p', r; P)$ denote the circle in P of radius r about p′.

If $\mathcal{C}$ is a circle and p_1, p_2, and p_3 are three points on $\mathcal{C}$, let $\gamma(p_1, p_2; p_3)$ denote the open arc of $\mathcal{C}$ that connects p_1 and p_2 and contains p_3, and let $\gamma^c(p_1, p_2; p_3)$ denote the open arc of $\mathcal{C}$ that connects p_1 and p_2 and does not contain p_3.

If $L \subseteq P$ is a line, let $\mathcal{S}(L, r; P)$ be the strip in P of width $2r$ about L; that is, $\mathcal{S}(L, r; P) = \{x \in P : d(x, L) \le r\}$, where d denotes Euclidean distance.

For $v \times w \ne 0$, let V(v, w, p) denote the planar cone $\{p + \alpha v + \beta w : \alpha, \beta \in \mathbb{R}, \alpha\beta \ge 0\}$. If V = V(v, w, p), then the complementary cone, V^c, is the cone V(−v, w, p). V and V^c partition the plane that contains them into two regions that intersect only at their boundaries. If ${}^t wv \ge 0$, then we say that V is acute. Either V or V^c (or possibly both) must be acute.

Let Π_P denote perpendicular projection onto P; that is, $\Pi_P : x \in \mathbb{R}^3 \to x - {}^t w(x - p)(w/\| w \|^2) \in P$. Note that Π_P is linear if and only if $0 \in P$, so that $\Pi_{P(w,0)} = \Pi_P - ({}^t wp)(w/\| w \|^2)$ is linear.

• *Spheres*

For $r > 0$ let $S(p, r)$ be the sphere about p of radius r.

For the sphere $S = S(p, r)$ let $H^-(w, q; S) = S \cap P^-(w, q)$, $G(w, q; S) = S \cap P(w, q)$, and $H^+(w, q; S) = S \cap P^+(w, q)$. Note that if $p \in P$, then H^- and H^+ are hemispheres bounded by the great circle G.

Let Π_S denote spherical projection onto S; that is, $\Pi_S : x \in \mathbb{R}^3 \backslash p \to p + r(x - p)/\| x - p \| \in S$.

• *Cylinders*

For $r > 0$, let $C(w, p, r)$ be the right circular cylinder with axis L(w, p) and radius r.

• *Cones*

For $r > 0$, let $V(w, p, r)$ be the right circular cone with vertex p, axis L(w, p), and such that if θ is the angle between any line in V and L(w, p), then $|\tan \theta| = r$.

For the cone $V = V(w, p, r)$, let $V^- = V^-(w, p, r) = V \cap P^-(w, p)$ and $V^+ = V^+(w, p, r) = V \cap P^+(w, p)$.

Let $V_i = V_i(w, p, r)$ be the union of the two regions inside V; that is,

$$V_i(w, p, r) = \left\{ x \in \mathbb{R}^3 \backslash p : \frac{({}^t w(x - p))^2}{\| w \|^2 \| (x - p) \|^2} > \frac{1}{r^2 + 1} \right\}$$
$$= \{x \in \mathbb{R}^3 \backslash p : |\tan \theta| < r\},$$

where θ is the angle between w and x − p,

and let $V_x = V_x(w, p, r)$ be the region outside V; that is, $V_x(w, p, r) = \mathbb{R}^3 \backslash (V \cup V_i)$.

Let $V_i^+ = V_i^+(w, p, r) = V_i \cap P^+(w, p)$ and $V_i^- = V_i^-(w, p, r) = V_i \cap P^-(w, p)$.

V_i^+ is the region inside the upper half-cone $V^+ \cup \mathrm{p}$, and V_i^- is the region inside the lower half-cone $V^- \cap \mathrm{p}$. Note that $V = V^- \cup V^+ \cup \mathrm{p}$ and that $\mathbb{R}^3 = V_i \cup V \cup V_x$, both as disjoint unions.

3. Planar projections

In this section, we first examine the projections onto a plane, which are simpler than those onto a sphere, as would be expected. We proceed here and in the next section in increasing order of complexity: that is, first with the projection of a plane, then a sphere, then a cylinder, and finally a cone.

- *Projection of a plane onto a plane*

Proposition 1
Let $\mathrm{P} = \mathrm{P}(\mathrm{v}, \mathrm{p})$ and $\mathrm{P}' = \mathrm{P}(\mathrm{v}', \mathrm{p}')$.

$$\Pi_{\mathrm{P}'}(\mathrm{P}) = \begin{cases} \mathrm{P}' & \text{if v is not perpendicular to v}', \\ \mathrm{L}(\mathrm{v} \times \mathrm{v}', \hat{\mathrm{p}}) & \text{if v is perpendicular to v}', \end{cases}$$

where $\hat{\mathrm{p}} = \mathrm{p} + {}^t\mathrm{v}'(\mathrm{p}' - \mathrm{p}) \| \mathrm{v}' \|^{-2} \mathrm{v}'$.

Proof Since ${}^t\mathrm{v}\hat{\mathrm{p}} = {}^t\mathrm{v}\mathrm{p}$ and ${}^t\mathrm{v}'\hat{\mathrm{p}} = {}^t\mathrm{v}'\mathrm{p}'$, when v is perpendicular to v′, it follows that $\hat{\mathrm{p}} \in \mathrm{P} \cap \mathrm{P}'$, and the claims are now easily verified. □

- *Projection of a sphere onto a plane*

Proposition 2
Let $\mathrm{P} = \mathrm{P}(\mathrm{w}, \mathrm{p})$ and let $S = S(\mathrm{q}, r)$. $\Pi_\mathrm{P}(S)$ is the closed disc $\Delta \subseteq \mathrm{P}$ about $\Pi_\mathrm{P}(\mathrm{q}) = \mathrm{q} - {}^t\mathrm{w}(\mathrm{q} - \mathrm{p})(\mathrm{w}/\| \mathrm{w} \|^2)$ of radius r. If Π_P is restricted to S, then it is twofold on the interior of Δ and onefold on $\partial\Delta$. (Interior and boundary are defined with respect to the natural topology of P.) See **Figure 1**.

Proof Obvious. □

- *Projection of a cylinder onto a plane*

Proposition 3
Let $\mathrm{P} = \mathrm{P}(\mathrm{w}, \mathrm{p})$ and let $C = C(\mathrm{v}, \mathrm{q}, r)$.

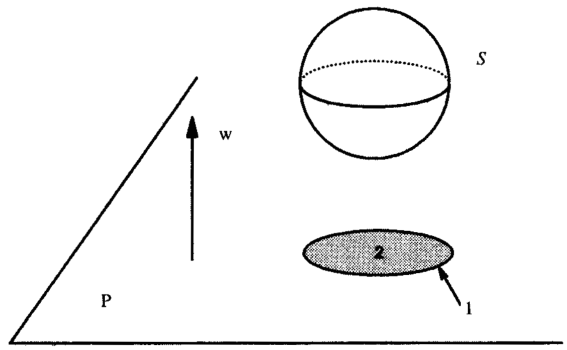


Figure 1
Projection of a sphere onto a plane.

$$\Pi_\mathrm{P}(C) = \begin{cases} \mathcal{C}(\Pi_\mathrm{P}(\mathrm{q}), r; \mathrm{P}) \\ \quad \text{if w and v are parallel,} \\ \mathcal{S}(\mathrm{L}(\| \mathrm{w} \|^2 \mathrm{v} - ({}^t\mathrm{wv})\mathrm{w}, \Pi_\mathrm{P}(\mathrm{q})), r; \mathrm{P}) \\ \quad \text{if w and v are not parallel.} \end{cases}$$

If Π_P is restricted to C, then in the first case each point of the circle is the image of a regula of C, and in the second case Π_P is onefold on the boundary of the image and twofold on the interior of the image. (Boundary and interior are defined with respect to the natural topology of P.) See **Figure 2**.

Proof If w and v are parallel, then $C = \{\mathrm{c} + t\mathrm{w} : \mathrm{c} \in \mathcal{C}(\Pi_\mathrm{P}(\mathrm{q}), r; \mathrm{P}) \text{ and } t \in \mathbb{R}\}$, and the result follows.

If w and v are not parallel, then

$$C = \left\{ r\left(\cos\theta \frac{\mathrm{v} \times \mathrm{w}}{\| \mathrm{v} \times \mathrm{w} \|} + \sin\theta \frac{\mathrm{v} \times \mathrm{w}}{\| \mathrm{v} \times \mathrm{w} \|} \times \frac{\mathrm{v}}{\| \mathrm{v} \|} \right) + \mathrm{q} + t\mathrm{v} : \theta \in [0, 2\pi) \text{ and } t \in \mathbb{R} \right\}.$$

If $\mathrm{Q} = \mathrm{P}(\mathrm{w}, 0)$, then, since $\mathrm{v} \times \mathrm{w} \perp \mathrm{w}$ and Π_Q is linear, it follows that

$$\begin{aligned} \Pi_\mathrm{P}(C) - \Pi_\mathrm{P}(q) &= \Pi_\mathrm{Q}(C) - \Pi_\mathrm{Q}(q) = \Pi_\mathrm{Q}(C - q) \\ &= \left\{ r\left(\cos\theta\, \Pi_\mathrm{Q}\left(\frac{\mathrm{v} \times \mathrm{w}}{\| \mathrm{v} \times \mathrm{w} \|} \right) + \sin\theta\, \Pi_\mathrm{Q}\left(\frac{\mathrm{v} \times \mathrm{w}}{\| \mathrm{v} \times \mathrm{w} \|} \times \frac{\mathrm{v}}{\| \mathrm{v} \|} \right) \right) + t\Pi_\mathrm{Q}(\mathrm{v}) : \theta \in [0, 2\pi) \text{ and } t \in \mathbb{R} \right\} \\ &= \left\{ r\cos\theta \frac{\mathrm{v} \times \mathrm{w}}{\| \mathrm{v} \times \mathrm{w} \|} + \frac{r\sin\theta}{\| \mathrm{v} \times \mathrm{w} \| \| \mathrm{v} \|} \Pi_\mathrm{Q}(\| \mathrm{v} \|^2 \mathrm{w} - ({}^t\mathrm{vw})\mathrm{v}) + t\Pi_\mathrm{Q}(\mathrm{v}) : \theta \in [0, 2\pi) \text{ and } t \in \mathbb{R} \right\} \\ &= \left\{ r\cos\theta \frac{\mathrm{v} \times \mathrm{w}}{\| \mathrm{v} \times \mathrm{w} \|} - \frac{r\sin\theta({}^t\mathrm{vw})}{\| \mathrm{v} \times \mathrm{w} \| \| \mathrm{v} \|} \Pi_\mathrm{Q}(\mathrm{v}) + t\Pi_\mathrm{Q}(\mathrm{v}) : \theta \in [0, 2\pi) \text{ and } t \in \mathbb{R} \right\} \\ &= \left\{ r\cos\theta \frac{\mathrm{v} \times \mathrm{w}}{\| \mathrm{v} \times \mathrm{w} \|} + t\Pi_\mathrm{Q}(\mathrm{v}) : \theta \in [0, 2\pi) \text{ and } t \in \mathbb{R} \right\}. \end{aligned}$$

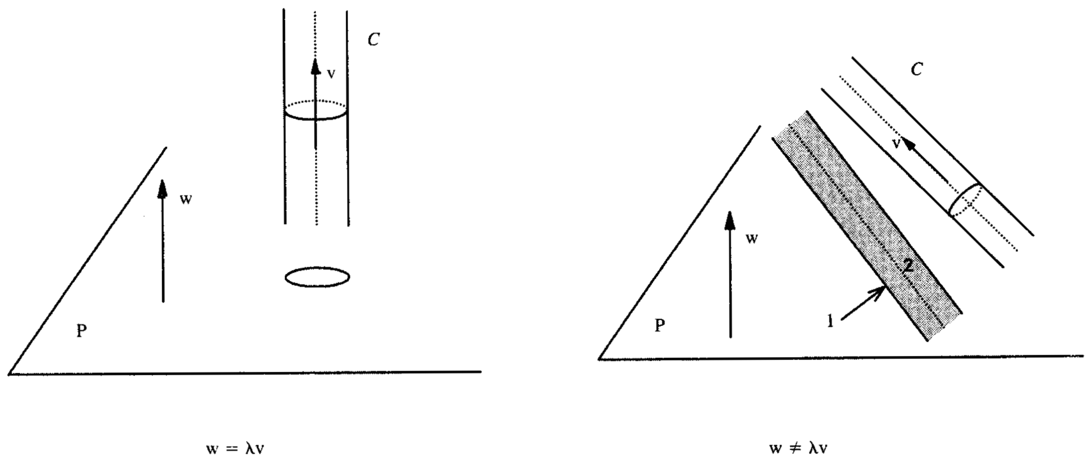


Figure 2

Projection of a cylinder onto a plane.

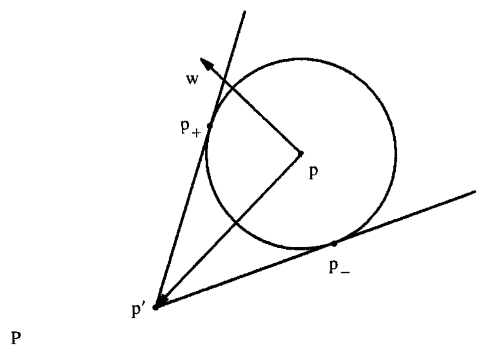


Figure 3

A circle with an exterior point and two points of tangency.

From this this result follows immediately, since $\Pi_Q(v) = v - ({}^t wv)(w/\| w \|^2)$, so that $v \times w \perp \Pi_Q(v)$. □

• *A technical lemma*

The following lemma and its two corollaries will be useful in the final planar projection we consider, and also in the section on spherical projections. The lemma would possibly seem simpler if expressed in terms of sin and cos, but would be more troublesome in symbolic manipulations later, so these are avoided here.

Lemma 1

Let $P = P(v, q)$. Let $\mathcal{C} = \mathcal{C}(p, r; P)$ and let $p' \in P$ be outside $\mathcal{C}$. Let

$$w = \frac{v \times (p' - p)}{\| v \times (p' - p) \|}.$$

Let

$$p_\pm = p + \frac{r^2}{\| p' - p \|^2}(p' - p) \pm r\sqrt{1 - \frac{r^2}{\| p' - p \|^2}}\, w.$$

The lines $L(p_- - p', p')$ and $L(p_+ - p', p')$ are tangent to $\mathcal{C}$ at p_- and p_+, respectively. They are the only lines tangent to $\mathcal{C}$ and containing p'. See **Figure 3**.

Proof One checks easily that $\| p_- - p \| = r$ and that $(p_- - p) \perp (p_- - p')$, which imply the claims for the first line. The same calculations work for the second line. That no more than two lines can share these properties is obvious. □

Corollary 1

Let $S = S(p, r)$ and p' be outside S. Let

$$V = V\left(p - p', p', \frac{r}{\sqrt{\|p - p'\|^2 - r^2}} \right).$$

V circumscribes S. No other lines through p′ are tangent to S.

Proof If one applies the lemma to each plane containing p and p′, and calculates the tangent of the angle between $L(p - p', p')$ and $L(p_\pm - p', p')$, then the corollary follows. □

Corollary 2

Let $P = P(v, q')$. Let $\mathcal{E} = \{p + r\cos\theta x + r'\sin\theta y : \theta \in [0, 2\pi)\}$ with x and y orthonormal be an ellipse in P, and let $p' \in P$ be outside $\mathcal{E}$. Let

$$p_\pm = p + \frac{p' - p}{\|S(p' - p)\|^2} \pm \sqrt{\frac{\|S(p' - p)\|^2 - 1}{\|S(p' - p)\|^4}} \left(\frac{-{}^t y(p' - p) r}{r'} x + \frac{{}^t x(p' - p) r'}{r} y \right),$$

where $S : q \in \mathbb{R}^3 \to ({}^t xq/r)x + ({}^t yq/r')y$. The lines $L(p_- - p', p')$ and $L(p_+ - p', p')$ are tangent to $\mathcal{E}$ at p_- and p_+, respectively. They are the only lines tangent to $\mathcal{E}$ and containing p′. If $p' - p \perp x$, then

$$p_\pm = p + r'^2 \frac{p' - p}{\|p' - p\|^2} \pm r \sqrt{1 - \frac{r'^2}{\|p' - p\|^2}}\, x.$$

Proof Since p′ is outside $\mathcal{E}$, then

$$\|S(p' - p)\|^2 = \frac{({}^t x(p' - p))^2}{r^2} + \frac{({}^t y(p' - p))^2}{r'^2} > 1,$$

so that $S(p' - p)$ is outside $\mathcal{C}(0, 1; P(0, v)) = \mathcal{C}$. Applying the lemma to $S(p' - p)$ and $\mathcal{C}$ yields points of tangency at

$$\hat{p}_\pm = \frac{S(p' - p)}{\|S(p' - p)\|^2} \sqrt{1 - \frac{1}{\|S(p' - p)\|^2}} \frac{v \times S(p' - p)}{\|v \times S(p' - p)\|}$$

$$= \frac{S(p' - p)}{\|S(p' - p)\|^2} \pm \sqrt{1 - \frac{1}{\|S(p' - p)\|^2}} \frac{\|v\| \left(\frac{-{}^t y(p' - p)}{r'} x + \frac{{}^t x(p' - p)}{r} y \right)}{\|v\| \|S(p' - p)\|}$$

$$= \frac{S(p' - p)}{\|S(p' - p)\|^2} + \sqrt{\frac{\|S(p' - p)\|^2 - 1}{\|S(p' - p)\|^4}} \left(\frac{-{}^t y(p' - p)}{r'} x + \frac{{}^t x(p' - p)}{r} y \right).$$

We now define the affine transformation $T : q \in \mathbb{R}^3 \to {}^t xqrx + {}^t yqr'y + p$. It is easy to see that $T(\mathcal{C}) = \mathcal{E}$ and that $T(S(p' - p)) = p'$. Since tangency is an affine invariant, it follows that $T(\hat{p}_\pm)$ must be the points of tangency for the lines through p′ and tangent to $\mathcal{E}$.

The final claim follows immediately, since if $p' - p \perp x$, then

$$\|S(p' - p)\|^2 = \frac{({}^t y(p - p))^2}{r'^2}$$

and ${}^t y(p' - p) = \pm \|p' - p\|$. □

Projection of a cone onto a plane

The final planar projection considered is that of a cone. The most obvious situation yielding a planar cone turns out to be the most difficult to describe precisely, while the other two cases are trivial to describe but easy to overlook.

Proposition 4

Let $P = P(v, q)$ and $V = V(w, p, r)$.

$$\Pi_P(V) = \begin{cases} P & \text{if } v + p \text{ is inside } V, \\ P \setminus L^\times & \text{if } v + p \in V, \\ \mathrm{V} & \text{if } v + p \text{ is outside } V, \end{cases}$$

where $L^\times = L^+(w \times v, \Pi_P(p)) \cup L^-(w \times v, \Pi_P(p))$, and V is the planar cone,

$$\mathrm{V} = \mathrm{V}(u_-, u_+, p + ({}^t v(q - p))\|v\|^{-2} v),$$

with $u_\pm = [\|w\|^2(\|w \times v\|^2 - r^2({}^t wv)^2)]^{1/2} v \times (w \times v) \pm r\|w\|^2\|v\|^2 w \times v$.

If Π_P is restricted to V, then in the first case it is onefold on $\Pi_P(p)$ and twofold elsewhere; in the second case a full line is mapped to $\Pi_P(p)$ and it is onefold elsewhere; and in the final case it is twofold on the interior of its image and onefold on the boundary of its image. (Interior and boundary are defined with respect to the natural topology of P.)

Proof If w is parallel or orthogonal to v, then the claims are obvious, so assume that $\|v \times w\| \neq 0 \neq {}^t wv$. This implies that

$$\Pi_P\left(\left\{p + \frac{w}{\| w \|} + r\left(\cos\theta \frac{v \times w}{\| v \times w \|} + \sin\theta \frac{(v \times w) \times w}{\| (v \times w) \times w \|}\right): \theta \in [0, 2\pi)\right\}\right)$$

is an ellipse, $\mathcal{E}$.

Since

$$V = \left\{p + t\left(\frac{w}{\| w \|} + r\left(\cos\theta \frac{v \times w}{\| v \times w \|} + \sin\theta \frac{(v \times w) \times w}{\| (v \times w) \times w \|}\right)\right): t \in \mathbb{R}, \theta \in [0, 2\pi)\right\},$$

the image of V is the union of the lines passing through $\Pi_P(p)$ and a point of $\mathcal{E}$. The three cases above correspond to $\Pi_P(p)$ inside, on, or outside $\mathcal{E}$, respectively. This observation and simple arguments lead immediately to the claims for the first two cases. For the third case, more detailed information about $\mathcal{E}$ is needed.

First, let us assume that $q = 0$, so that Π_P is linear, and

$$\mathcal{E} = \left\{\Pi_P(p) + \Pi_P\left(\frac{w}{\| w \|}\right) + r\left(\cos\theta \frac{v \times w}{\| v \times w \|} + \sin\theta \frac{\Pi_P((v \times w) \times w)}{\| (v \times w) \times w \|}\right): \theta \in [0, 2\pi)\right\}$$

$$= \left\{\Pi_P(p) + \Pi_P\left(\frac{w}{\| w \|}\right) + r\left(\cos\theta \frac{v \times w}{\| v \times w \|} + \sin\theta \frac{\Pi_P({}^t wvw - \| w \|^2 v)}{\| (v \times w) \times w \|}\right): \theta \in [0, 2\pi)\right\}$$

$$= \left\{\Pi_P(p) + \frac{w}{\| w \|} - \frac{{}^t vwv}{\| w \| \| v \|^2} + r\left(\cos\theta \frac{v \times w}{\| v \times w \|} + \sin\theta \frac{{}^t wvw - \frac{({}^t wv)^2 v}{\| v \|^2}}{\| (v \times w) \times w \|}\right): \theta \in [0, 2\pi)\right\}.$$

Since $\| \| v \|^2 w - {}^t wvv \|^2 = \| v \|^4 \| w \|^2 - 2 \| v \|^2 ({}^t wv)^2 + \| v \|^2 ({}^t wv)^2 = \| v \|^2 (\| v \|^2 \| w \|^2 - ({}^t wv)^2) = \| v \|^2 \| v \times w \|^2$, $\mathcal{E}$ has a semimajor axis of length r and a semiminor axis of length $r | {}^t wv | / (\| v \| \| w \|)$. The distance from the center of $\mathcal{E}$ to $\Pi_P(p)$ is

$$\left\| \Pi_P\left(\frac{w}{\| w \|}\right) \right\| = \left\| \frac{w}{\| w \|} - \frac{{}^t vwv}{\| w \| \| v \|^2} \right\| = \sqrt{1 - \frac{({}^t vw)^2}{\| w \|^2 \| v \|^2}} = \| v \times w \| / (\| w \| \| v \|).$$

Since $\Pi_P(w) \perp v \times w$, Corollary 2 implies that the two lines through $\Pi_P(p)$ and

$$p_\pm = \Pi_P(p) + \Pi_P\left(\frac{w}{\| w \|}\right) - (r | {}^t wv | / (\| v \| \| w \|))^2 \frac{\Pi_P\left(\frac{w}{\| w \|}\right)}{\left\| \Pi_P\left(\frac{w}{\| w \|}\right) \right\|^2} \pm r \sqrt{1 - \frac{(r | {}^t wv | / (\| v \| \| w \|))^2}{\left\| \Pi_P\left(\frac{w}{\| w \|}\right) \right\|^2}} \frac{v \times w}{\| v \times w \|}$$

$$= \Pi_P(p) + \left(1 - \frac{r^2 ({}^t wv)^2}{\| v \times w \|^2}\right) \frac{\Pi_P(w)}{\| w \|} \pm r \sqrt{1 - \frac{r^2 ({}^t wv)^2}{\| v \times w \|^2}} \frac{v \times w}{\| v \times w \|}$$

are tangent to $\mathcal{E}$ at $p_\pm$. This establishes the result for $q = 0$, and since

$$\Pi_P(z) = \Pi_{P(v,0)}(z) + \frac{{}^t vq}{\| v \|^2} v,$$

the general result follows. □

4. Spherical projections

Now we turn to the spherical projections. The first two are straightforward, but those for cylinders and cones are more interesting.

- *Projection of a plane onto a sphere*

Proposition 5

Let $P = P(v, p)$ and $S' = S(p', r')$.

$$\Pi_{S'}(P) = \begin{cases} H^\sigma(v, p'; S') & \text{if } p' \text{ is not in } P, \\ G(v, p'; S') & \text{if } p' \text{ is in } P, \end{cases}$$

where $\sigma = \text{sgn}({}^t v(p - p'))$.

Proof Obvious. □

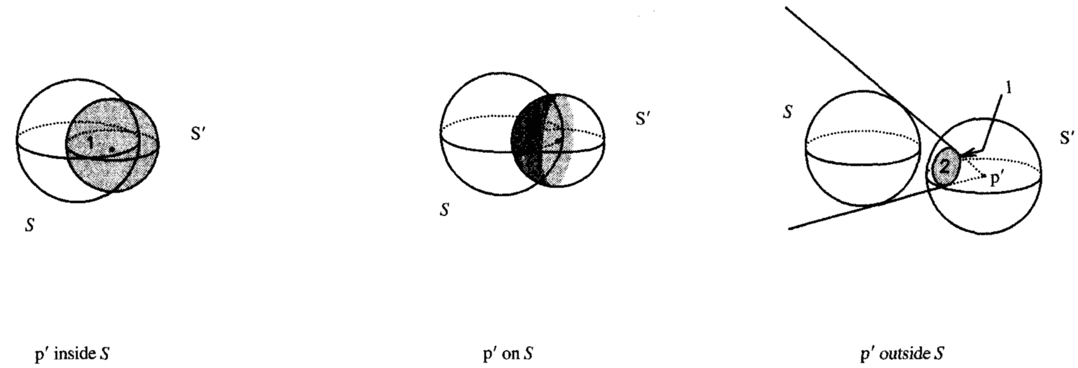


Figure 4

Projection of a sphere onto a sphere.

• *Projection of a sphere onto a sphere*

Proposition 6
Let $S = S(\mathrm{p}, r)$ and $S' = S(\mathrm{p}', r')$.

$$\Pi_{S'}(S) = \begin{cases} S' & \text{if } \mathrm{p}' \text{ is inside } S, \\ H^{+}(\mathrm{p} - \mathrm{p}', \mathrm{p}'; S') & \text{if } \mathrm{p}' \text{ is on } S, \\ \overline{H}^{+}\left(\mathrm{p} - \mathrm{p}', \mathrm{p}' + r' \sqrt{\| \mathrm{p} - \mathrm{p}' \|^2 - r^2}\, \dfrac{\mathrm{p} - \mathrm{p}'}{\| \mathrm{p} - \mathrm{p}' \|^2}; S'\right) & \text{if } \mathrm{p}' \text{ is outside } S. \end{cases}$$

If $\Pi_{S'}$ is restricted to $S \backslash \mathrm{p}'$, then in the first two cases it is onefold everywhere, while in the third case it is twofold on the interior of its image and onefold on the boundary of its image. (Interior and boundary are defined with respect to the natural topology of S'.) See **Figure 4**.

Proof If p′ is inside S, the claim is obvious. When p′ is on S, by considering the hyperplane of support for S at p', the claim is easily verified. The last case follows from Corollary 1 and simple trigonometry. □

• *Projection of a cylinder onto a sphere*
The proposition describing the projection of a cylinder onto a sphere depends on two technical lemmas.

Lemma 2
The point $\hat{\mathrm{p}} = {}^{t}\mathrm{w}(\mathrm{p}' - \mathrm{p})(\mathrm{w}/\| \mathrm{w} \|^2) + \mathrm{p}$ minimizes the distance from a point on L(w, p) to the point p′.

Proof $(\hat{\mathrm{p}} - \mathrm{p}') \perp \mathrm{w}$. □

Lemma 3
Let $S' = S(\mathrm{p}', r')$ and $\mathrm{L} = \mathrm{L}(\mathrm{w}, \mathrm{p})$ be a line such that $\mathrm{p}' \notin \mathrm{L}$. $\Pi_{S'}(\mathrm{L})$ is the "open" half of the great circle of S' determined by the plane containing L and p′ that runs between $\mathrm{p}' - r'(\mathrm{w}/\| \mathrm{w} \|)$ and $\mathrm{p}' + r'(\mathrm{w}/\| \mathrm{w} \|)$ and is "closest" to L. More precisely, if $\hat{\mathrm{p}}$ minimizes the distance from L to p′, then $\Pi_{S'}(\mathrm{L}) = G((\mathrm{p} - \mathrm{p}') \times \mathrm{w}, \mathrm{p}'; S') \cap H^{+}(\hat{\mathrm{p}} - \mathrm{p}', \mathrm{p}'; S')$. $\Pi_{S'}$ restricted to L is onefold. See **Figure 5**.

Proof Let $\mathrm{P} = \mathrm{P}((\hat{\mathrm{p}} - \mathrm{p}') \times \mathrm{w}, \mathrm{p}')$. Since clearly $\mathrm{L} \subseteq \mathrm{P}$ and $\mathrm{p}' \subseteq \mathrm{P}$, it follows directly from the definition of $\Pi_{S'}$ that $\Pi_{S'}(\mathrm{L}) \subseteq \mathrm{P} \cap S'$. The claims now follow from the analogous, but obvious, planar result concerning the projection of a line onto a circle whose center is not on the line. □

If C is a cylinder and V is a cone, we could use Proposition 4 to obtain $\Pi_{\mathrm{P}}(V)$, for P orthogonal to the axis of C, and then segment the parameter domain of the ruled

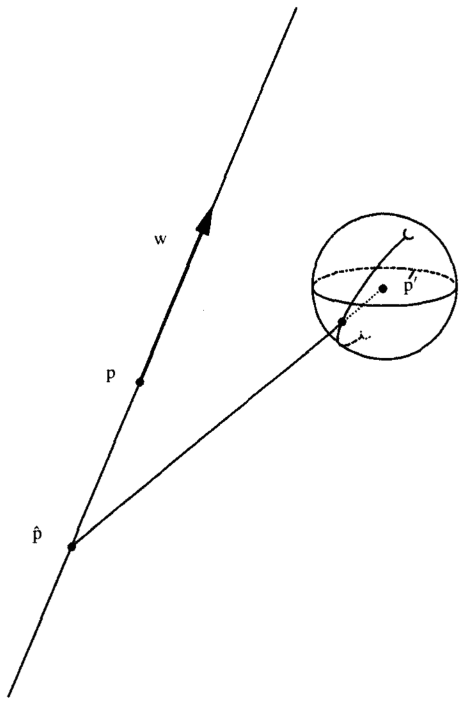


Figure 5

Projection of a line onto a sphere.

surface, C, by intersecting circle $\mathcal{C} = C \cap P$ with $\Pi_P(V)$. Alternatively, we could use the following proposition to project C onto a sphere, S, whose center coincides with the vertex of V, and segment the parameter domain of V by intersecting circle $\mathcal{C}' = V \cap S$ with this projection. Thus, as the means to aid in cone–cylinder intersection problems, either Proposition 4 or the following alone would suffice.

Proposition 7

Let $C = C(\mathrm{w}, \mathrm{p}, r)$ and $S' = S(\mathrm{p}', r')$.

$$\Pi_{S'}(C) = \begin{cases} S' \backslash \mathcal{P} & \text{if } \mathrm{p}' \text{ is inside } C, \\ \mathcal{P} \cup H^+(\hat{\mathrm{p}} - \mathrm{p}', \mathrm{p}'; S') & \text{if } \mathrm{p}' \text{ is on } C, \\ \bar{H}^+(\mathrm{n}_-, \mathrm{p}'; S') & \\ \quad \cap \bar{H}^+(\mathrm{n}_+, \mathrm{p}'; S') \backslash \mathcal{P} & \text{if } \mathrm{p}' \text{ is outside } C, \end{cases}$$

where

$$\mathcal{P} = \left\{ \mathrm{p}' \pm r' \frac{\mathrm{w}}{\|\mathrm{w}\|} \right\}, \qquad \hat{\mathrm{p}} = {}^t\mathrm{w}(\mathrm{p}' - \mathrm{p}) \frac{\mathrm{w}}{\|\mathrm{w}\|^2} + \mathrm{p},$$

$$\mathrm{n}_\pm = \hat{\mathrm{p}} - \mathrm{p}' \pm \left[\frac{\|\hat{\mathrm{p}} - \mathrm{p}'\|^2 - r^2}{r^2 \|\mathrm{w}\|^2} \right]^{1/2} \mathrm{w} \times (\hat{\mathrm{p}} - \mathrm{p}').$$

If $\Pi_{S'}$ is restricted to $C \backslash \mathrm{p}'$, then in the first case it is onefold everywhere; in the second case it is onefold on $H^+(\hat{\mathrm{p}} - \mathrm{p}', \mathrm{p}'; S')$, while it maps the half-line $[\mathrm{p}' + t\,\mathrm{w}: t > 0\}$ onto $\mathrm{p}' + r'(\mathrm{w}/\|\mathrm{w}\|)$ and the half-line $\{\mathrm{p}' + t\,\mathrm{w}: t < 0\}$ onto $\mathrm{p}' - r'(\mathrm{w}/\|\mathrm{w}\|)$; and in the third case it is twofold on the interior of $\Pi_{S'}(C)$ and onefold on $\partial\, \Pi_{S'}(C) \backslash \mathcal{P}$. (Interior and boundary are defined with respect to the natural topology of S'.) See **Figure 6**.

Proof Let p′ be inside C, L = L(v, p′), and P be any plane containing L and p′ + w. $\mathrm{P} \cap C$ will be two lines L_1 and L_2, neither of which will coincide with L. $\mathrm{L} \cap C = \mathrm{L} \cap (\mathrm{L}_1 \cup \mathrm{L}_2)$ will be two points if L is not parallel to C, and empty if it is. This implies the result in the first case.

Let p′ be on C. Let P be the tangent plane to C at p′. It is easily seen that $\hat{\mathrm{p}} - \mathrm{p}' \perp \mathrm{P}$. Since P is a hyperplane of support of C at p′, it follows that if ${}^t\mathrm{v}(\hat{\mathrm{p}} - \mathrm{p}') < 0$, then $\mathrm{L}^+(\mathrm{v}, \mathrm{p}') \cap C = \varnothing$. $\mathrm{P} \cap C = \mathrm{L}(\mathrm{w}, \mathrm{p}')$. If ${}^t\mathrm{v}(\hat{\mathrm{p}} - \mathrm{p}') > 0$, since P is the tangent plane to C at p′, then $\mathrm{L}^+(\mathrm{v}, \mathrm{p}')$ must begin inside C, so that by the first part of this proof the closure of $\mathrm{L}^+(\mathrm{v}, \mathrm{p}')$ intersects C twice, once at p′ and once at another point. Since distinct rays share the initial point p′, these points of intersection cannot coincide.

Now let p′ be outside C. Let P = P(w, p′). $\mathrm{P} \cap C$ is a circle $\mathcal{C}$ of radius r about $\hat{\mathrm{p}}$. Lemma 1 implies that $\Pi_{S'}(\mathcal{C})$ is the arc of $G(\mathrm{w}, \mathrm{p}'; S')$ between $r'(\mathrm{p}_- - \mathrm{p}')/\|\mathrm{p}_- - \mathrm{p}'\|$ and $r'(\mathrm{p}_+ - \mathrm{p}')/\|\mathrm{p}_+ - \mathrm{p}'\|$ containing $\Pi_{S'}(\hat{\mathrm{p}})$, and that $\Pi_{S'}$ is onefold on the two endpoints and twofold on the interior of the arc. Each line of C passes through one point of circle $\mathcal{C}$ and is parallel to w. Lemma 3 implies that each line is mapped onto an open half-circle emanating from $\mathrm{p}' - r'\mathrm{w}/\|\mathrm{w}\|$ and terminating at $\mathrm{p}' + r'\mathrm{w}/\|\mathrm{w}\|$. Since two circles coincide if and only if they coincide at three points, the half-circles agree on $\Pi_{S'}(\mathcal{C})$ if and only if they coincide everywhere.

Since by Lemma 1

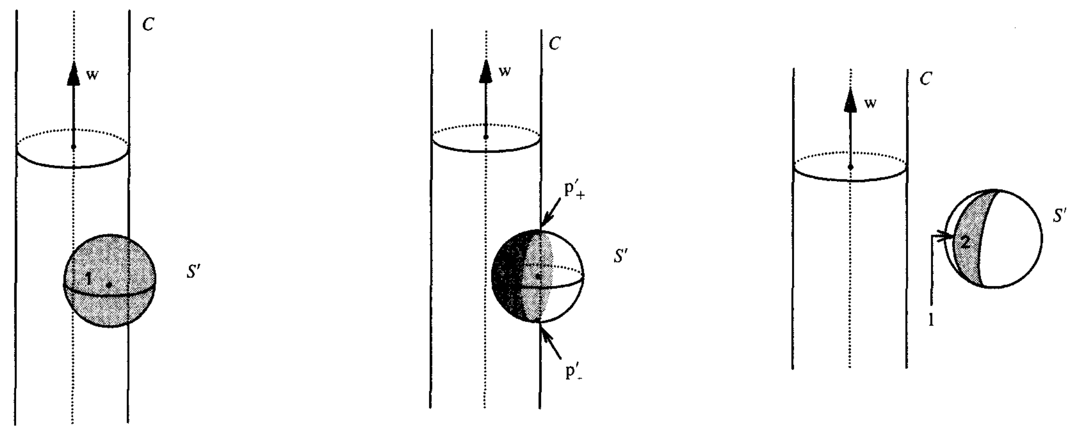
$$\mathrm{p}_\pm - \mathrm{p}' = \left(\frac{r^2}{\|\mathrm{p}' - \hat{\mathrm{p}}\|^2} - 1 \right)(\mathrm{p}' - \hat{\mathrm{p}})$$
$$\pm r \sqrt{1 - \frac{r^2}{\|\mathrm{p}' - \hat{\mathrm{p}}\|^2}} \frac{\mathrm{w} \times (\mathrm{p}' - \hat{\mathrm{p}})}{\|\mathrm{w} \times (\mathrm{p}' - \hat{\mathrm{p}})\|},$$

then

$${}^t(\mathrm{p}_\pm - \mathrm{p}')\mathrm{n}_\pm = \|\mathrm{p}' - \hat{\mathrm{p}}\|^2 - r^2$$
$$- \frac{\|\mathrm{p}' - \hat{\mathrm{p}}\|^2 - r^2}{\|\mathrm{p}' - \hat{\mathrm{p}}\| \, \|\mathrm{w}\|} \frac{\|\mathrm{w}\|^2 \|\mathrm{p}' - \hat{\mathrm{p}}\|^2}{\|\mathrm{w}\| \, \|\mathrm{p}' - \hat{\mathrm{p}}\|} = 0.$$

It is also clear that ${}^t\mathrm{n}_\pm(\mathrm{w}) = 0$. Thus $\mathrm{n}_\pm$ is perpendicular to the plane containing p′, $\Pi_{S'}(\mathrm{p}_\pm)$, and $\mathrm{p}' - r'(\mathrm{w}/\|\mathrm{w}\|)$. Since

$${}^t(\mathrm{p} - \mathrm{p}')\mathrm{n}_\pm = {}^t(\mathrm{p} - \mathrm{p}')(\hat{\mathrm{p}} - \mathrm{p}')$$
$$= -\frac{({}^t\mathrm{w}(\mathrm{p}' - \mathrm{p}))^2}{\|\mathrm{w}\|^2} + \|\mathrm{p}' - \mathrm{p}\|^2,$$



p′ inside C p′ on C p′ outside C

Figure 6

Projection of a cylinder onto a sphere.

the Cauchy–Schwartz inequality implies that $\Pi_{S'}(p) \in \bar{H}^+(n_-, p'; S') \cap \bar{H}^+(n_+, p'; S')$. This implies the claim for the final case. □

- *Two technical lemmas*

The final projection, that of a cone onto a sphere, is by far the most complicated. Four cases need to be distinguished which differ only by the location of the center of the sphere relative to the cone. The four occur when the center coincides with the vertex of the cone, when the center is on the cone, when the center is inside the cone, and when the center is outside the cone. Only in the last case is the image difficult to describe precisely (see Figure 9, shown later). The two lemmas which precede the proposition make the description of the final case easier. Each pertains to circles having special properties with respect to two antipodal circles and an exterior point on a sphere.

Lemma 4

Let $S = S(p, r)$. Let $\mathcal{C}^+ = G(v, p + v; S)$ and let $\mathcal{C}^- = G(v, p - v; S)$ with $0 < \| v \| < r$. Let $q \in H^-(v, p + v; S) \cap H^-(-v, p - v; S)$; that is, q belongs to the region of S between the two antipodal spherical caps bounded by $\mathcal{C}^+$ and $\mathcal{C}^-$. Finally, define

$$p_\pm^+ = p + (r^2 \| v \|^2 - ({}^t v(q - p))^2)^{-1}\{r^2 \| v \|^{-2}(\| v \|^4 - ({}^t v(q - p))^2)v + ({}^t v(q - p))(r^2 - \| v \|^2)(q - p)$$
$$\pm [r^2 \| v \|^{-2}(r^2 - \| v \|^2)(\| v \|^4 - ({}^t v(q - p))^2)]^{1/2}(q - p) \times v\},$$

$$p_+^- = p - (p_-^+ - p), \qquad p_-^- = p - (p_+^+ - p),$$

and

$$n_\pm = \mp(q - p) \times v + \sqrt{\frac{\| v \|^2(r^2 - \| v \|^2)}{r^2(\| v \|^4 - ({}^t v(q - p))^2)}}\,((q - p) \times v) \times (q - p).$$

The only two great circles of S passing through q and tangent to $\mathcal{C}^+$ or $\mathcal{C}^-$ are $G^+ = G(n_+, q; S)$ and $G^- = G(n_-, q; S)$. G^+ intersects $\mathcal{C}^+$ only at p_+^+ and intersects $\mathcal{C}^-$ only at p_-^-, and G^- intersects $\mathcal{C}^+$ only at p_-^+ and intersects $\mathcal{C}^-$ only at p_+^-. $p_+^+ \in H^+(n_-, q; S)$, and $p_-^+ \in H^+(n_+, q; S)$. See **Figure 7**.

Proof If a great circle G is tangent to $\mathcal{C}^+$ at $\hat{p}$, then $p - (\hat{p} - p)$ must lie in the plane which determines G, so that it also is on G. The symmetry of the sphere and antipodal circles then implies that G must also be tangent to $\mathcal{C}^-$ at the point antipodal to $\hat{p}$. Any two circles which are tangent at one point and which share a different point must coincide, so that no great circle tangent to $\mathcal{C}^+$ can intersect it again, since by definition $\mathcal{C}^+$ is not a great circle. That only two great circles may pass through q and be tangent to $\mathcal{C}^+$ is clear.

If ${}^t v(q - p) = 0$, since most terms vanish or simplify greatly, the claims are readily verified. Thus we assume that ${}^t v(q - p) \neq 0$.

Let

$$p' = p + \frac{\| v \|^2}{{}^t v(q - p)} (q - p).$$

Since ${}^t vp' = {}^t vp + \| v \|^2 = {}^t v(p + v)$, $p' \in P(v, p + v) \supseteq \mathcal{C}^+$. Since $\mathcal{C}^+$ is on S, and its center is $p + v$, its radius must equal $\sqrt{r^2 - \| v \|^2}$. Since

$$p' - (p + v) = \left(p + \frac{\| v \|^2}{{}^t v(q - p)} (q - p)\right) - (p + v) = \frac{\| v \|^2 (q - p) - {}^t v(q - p)v}{{}^t v(q - p)},$$

if we define

$$p_\pm^+ = p + v + \frac{{}^t v(q - p)(r^2 - \| v \|^2)}{\| \| v \|^2 (q - p) - {}^t v(q - p)v \|^2} (\| v \|^2 (q - p) - {}^t v(q - p)v)$$

$$\pm \sqrt{r^2 - \| v \|^2} \sqrt{1 - \frac{({}^t v(q - p))^2 (r^2 - \| v \|^2)}{\| \| v \|^2 (q - p) - {}^t v(q - p)v \|^2}} \frac{(q - p) \times v}{\| (q - p) \times v \|},$$

then by Lemma 1 p_+^+ is the point of $\mathcal{C}^+$ at which the line L^+ through p_+^+ and p' is tangent to $\mathcal{C}^+$, and p_-^+ is the point of $\mathcal{C}^+$ at which the line L^- through p_-^+ and p' is tangent to $\mathcal{C}^+$. If P^+ is the plane containing p and L^+, and P^- is the plane containing p and L^-, then by construction $q \in P^+$ and P^+ is tangent to $\mathcal{C}^+$ at p_+^+, and $q \in P^-$ and P^- is tangent to $\mathcal{C}^+$ at p_-^+. The geodesics determined by these planes thus satisfy the claims of the lemma.

We simplify the expressions of $p_\pm^+$ by recognizing that $\| q - p \| = r$, since $q \in S$, and by using the following formulas for cross products of arbitrary vectors a and b:

- $\| (a \times b) \times b \| = \| a \times b \| \| b \|$.
- $\| a \times b \|^2 = \| a \|^2 \| b \|^2 - ({}^t ab)^2$.
- $(a \times b) \times b = {}^t bab - \| b \|^2 a$.

Thus,

$$p_\pm^+ = p + v - \frac{{}^t v(q - p)(r^2 - \| v \|^2)}{\| ((q - p) \times v) \times v \|^2} ((q - p) \times v) \times v$$

$$\pm \sqrt{r^2 - \| v \|^2} \sqrt{1 - \frac{({}^t v(q - p))^2 (r^2 - \| v \|^2)}{\| ((q - p) \times v) \times v \|^2}} \frac{(q - p) \times v}{\| (q - p) \times v \|}$$

$$= p + v - \frac{{}^t v(q - p)(r^2 - \| v \|^2)}{\| v \|^2 \| (q - p) \times v \|^2} ((q - p) \times v) \times v$$

$$\pm \sqrt{r^2 - \| v \|^2} \sqrt{\frac{\| v \|^2 \| (q - p) \times v \|^2 - ({}^t v(q - p))^2 (r^2 - \| v \|^2)}{\| v \|^2}} \frac{(q - p) \times v}{\| (q - p) \times v \|^2}$$

$$= p + v - \frac{{}^t v(q - p)(r^2 - \| v \|^2)}{\| v \|^2 (\| v \|^2 r^2 - ({}^t v(q - p))^2)} ({}^t v(q - p)v - \| v \|^2 (q - p))$$

$$\pm \sqrt{\frac{(r^2 - \| v \|^2) r^2 (\| v \|^4 - ({}^t v(q - p))^2)}{\| v \|^2}} \frac{(q - p) \times v}{\| v \|^2 r^2 - ({}^t v(q - p))^2}.$$

Collecting terms and applying simple algebra yield the desired formulas for p_+^+ and p_-^+ of the lemma.

If we let

$$\alpha = {}^t((q-p)\times v)\,\frac{1}{r^2\|v\|^2 - ({}^tv(q-p))^2}\sqrt{\frac{(r^2-\|v\|^2)r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2}}\,(q-p)\times v$$

$$= \sqrt{\frac{(r^2-\|v\|^2)r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2}}\,\frac{\|(q-p)\times v\|^2}{r^2\|v\|^2-({}^tv(q-p))^2}$$

$$= \sqrt{\frac{(r^2-\|v\|^2)r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2}}\,\frac{r^2\|v\|^2-({}^tv(q-p))^2}{r^2\|v\|^2-({}^t(q-p))^2}$$

$$= \sqrt{\frac{(r^2-\|v\|^2)r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2}}, \text{ and}$$

$$\beta = {}^t\left(\frac{r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2(r^2\|v\|^2-({}^tv(q-p))^2)}\,v\right)\left(\sqrt{\frac{\|v\|^2(r^2-\|v\|^2)}{r^2(\|v\|^4-({}^tv(q-p))^2)}}\,((q-p)\times v)\times(q-p)\right)$$

$$= \sqrt{\frac{(r^2-\|v\|^2)r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2}}\,\frac{{}^tv(((q-p)\times v)\times(q-p))}{r^2\|v\|^2-({}^tv(q-p))^2}$$

$$= -\sqrt{\frac{(r^2-\|v\|^2)r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2}}\,\frac{{}^tv({}^tv(q-p)(q-p)-r^2v)}{r^2\|v\|^2-({}^tv(q-p))^2}$$

$$= -\sqrt{\frac{(r^2-\|v\|^2)r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2}}\,\frac{({}^tv(q-p))^2-r^2\|v\|^2}{r^2\|v\|^2-({}^tv(q-p))^2}$$

$$= \sqrt{\frac{(r^2-\|v\|^2)r^2(\|v\|^4-({}^tv(q-p))^2)}{\|v\|^2}},$$

then $\alpha = \beta > 0$. In terms of these values it follows that ${}^tn_+(p_+^+ - p) = -\alpha + \beta = 0$, ${}^tn_-(p_-^+ - p) = -\alpha + \beta = 0$, ${}^tn_+(p_-^+ - p) = \alpha + \beta > 0$, and ${}^tn_-(p_+^+ - p) = \alpha + \beta > 0$. The final claims now follow, since it is obviously also true that ${}^tn_+(q-p) = 0$, and ${}^tn_-(q-p) = 0$. □

Lemma 5

Let $k = -{}^tv(q-p)r^2v + \|v\|^4(q-p)$. Let S, p, q, v, r, $p_\pm^+$, and $p_\pm^-$ be as in Lemma 4. p_+^+, p_-^+, p_+^-, and p_-^- belong to the great circle, $\mathcal{B} = G(k, p; S)$. $q \in H^+(k, p; S)$.

See **Figure 8**.

Proof Since

$${}^tk(p_+^+ - p) = {}^t(-{}^tv(q-p)r^2v + \|v\|^4(q-p))\,\frac{1}{\|v\|^2(r^2\|v\|^2-({}^tv(q-p))^2)}$$

$$[r^2(\|v\|^4-({}^tv(q-p))^2)v + \|v\|^{2}\,{}^tv(q-p)(r^2-\|v\|^2)(q-p)]$$

$$= \frac{r^{2}\,{}^tv(q-p)}{r^2\|v\|^2-({}^tv(q-p))^2}\,[-(\|v\|^4-({}^tv(q-p))^2)r^2 + (\|v\|^4-({}^tv(q-p))^2)\|v\|^2$$

$$-(r^2-\|v\|^2)({}^tv(q-p))^2 + (r^2-\|v\|^2)\|v\|^4]$$

$$= \frac{r^{2}\,{}^tv(q-p)}{r^2\|v\|^2-({}^tv(q-p))^2}\,[-\|v\|^4r^2 + ({}^tv(q-p))^2r^2 + \|v\|^6 - ({}^tv(q-p))^2\|v\|^2$$

$$-r^2({}^tv(q-p))^2 + \|v\|^2({}^tv(q-p))^2 + r^2\|v\|^4 - \|v\|^6] = 0,$$

then k is perpendicular to $(p_+^+ - p)$. Since $(p_+^+ - p)$ differs from $(p_-^+ - p)$ only by a term that is perpendicular to k, then k is also perpendicular to $(p_-^+ - p)$. This implies that p_+^+ and p_-^+ belong to $\mathscr{B}$. Since p_+^- and p_-^- are antipodal to p_+^+ and p_-^+, they also must belong to $\mathscr{B}$.

If θ is the angle between $q - p$ and v, then the constraints on the choice of q imply that

$$|r\cos(\theta)| < \|v\|, \text{ so that } \|v\|^2 \pm {}^t v(q-p)$$
$$= \|v\|(\|v\| \pm r\cos(\theta)) > 0.$$

Thus,

$${}^t(-{}^t v(q-p)r^2 v + \|v\|^4(q-p))(q-p)$$
$$= -({}^t v(q-p))^2 r^2 + r^2\|v\|^4$$
$$= r^2(\|v\|^2 - {}^t v(q-p))(\|v\|^2 + {}^t v(q-p)) > 0,$$

so that by definition $q \in H^+(k, p; S)$. □

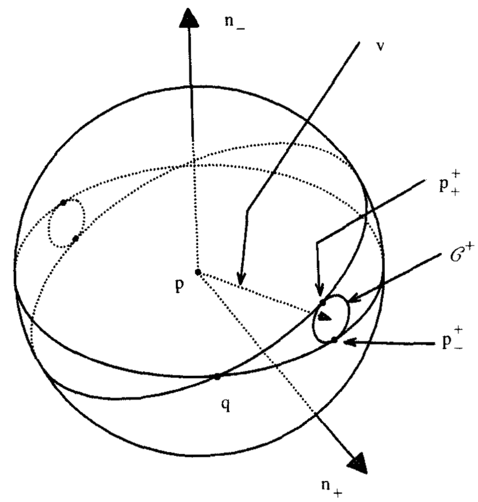


Figure 7

The two tangential great circles containing q.

• *Projection of a cone onto a sphere*

We are now ready for the final projection. The four previously mentioned cases—center and vertex coincide, center on cone, center inside cone, and center outside cone—are shown to have as images two antipodal circles, a hemisphere minus a spherical cap plus the antipodal cap, the sphere minus a spherical cap, and an hourglass-like region. In stating these results, care is required to avoid a notational nightmare, but each of the component parts in this version is nearly rational, involving at most one root.

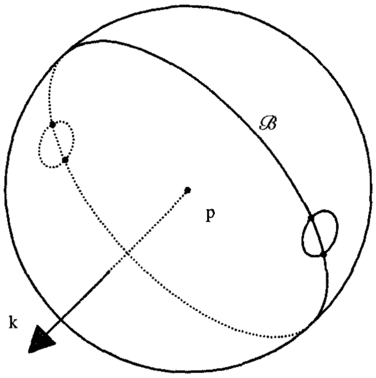


Figure 8

The great circle containing the four points of tangency.

Proposition 8

Let $V = V(w, p, r)$ and $S' = S(p', r')$.

If we define

$$\sigma = \operatorname{sgn}({}^t w(p - p')),$$

$$q^+ = \Pi_{S'}(p) = p' + r'\frac{p - p'}{\|p - p'\|},$$

$$q^- = p' - r'\frac{p - p'}{\|p - p'\|},$$

$$w^+ = p' + \frac{r'}{\sqrt{r^2 + 1}}\frac{w}{\|w\|},$$

$$w^- = p' - \frac{r'}{\sqrt{r^2 + 1}}\frac{w}{\|w\|},$$

$$w^\times = (w \times (p - p')) \times (p - p')$$
$$= {}^t w(p - p')(p - p') - \|p - p'\|^2 w,$$

$$k = -{}^t w(p - p')(r^2 + 1)w + \|w\|^2(p - p'),$$

$$n_\pm = \mp(p - p') \times w + \sqrt{\frac{\|w\|^2 r^2}{\|w\|^2\|p - p'\|^2 - (r^2 + 1)({}^t w(p - p'))^2}}\,((p - p') \times w) \times (p - p'),$$

and $\mathcal{V}^{\pm} = [\bar{H}^{\pm}(n_+, q^+; S') \cap \bar{H}^{\pm}(n_-, q^+; S') \cap H^+(k, p'; S')] \cup H^{\pm}(w, w^{\pm}; S')$,

$$\text{then}\quad \Pi_{S'}(V) = \begin{cases} G(w, w^+; S') \cup G(w, w^-; S') & \text{if } p' = p, \\ H^{\sigma}(w^{\times}, p'; S') \cap H^{\sigma}(w, w^{-\sigma}; S') \cup H^{\sigma}(w, w^{\sigma}; S') \cup q^{\pm} & \text{if } p' \text{ is on } V, \text{ but } p' \neq p, \\ H^{\sigma}(w, w^{-\sigma}; S') & \text{if } p' \text{ is inside } V, \\ \mathcal{V}^+ \cup \mathcal{V}^- & \text{if } p' \text{ is outside } V. \end{cases}$$

Let $\Pi_{S'}$ be restricted to $V \backslash p'$.

In the first case, if y belongs to the image of V, then $\Pi_{S'}(L^+(y - p', p')) = y$. In the second case $\Pi_{S'}(L^+(\pm(p - p'), p')) = q^{\pm}$, and is onefold elsewhere. In the third case it is twofold on $H^{\sigma}(w, w^{\sigma}; S')\backslash q^+$ and onefold elsewhere. In the final case $\Pi_{S'}$ is onefold on $H^+(w, w^+; S') \cup H^-(w, w^-; S')$ and on the boundary of $\Pi_{S'}(V)\backslash[H^+(w, w^+; S') \cup H^-(w, w^-; S')]$ and is twofold elsewhere. (Interior and boundary are defined with respect to the natural topology of S'.) See **Figure 9**.

Proof Since for any $y \in G(w, w^+; S') \cup G(w, w^-; S')$,

$$\frac{\sqrt{\| y - p' \|^2 - \left\| \left(p' \pm \frac{r'}{\sqrt{r^2 - 1}} \frac{w}{\| w \|}\right) - p' \right\|^2}}{\left\| \left(p' \pm \frac{r'}{\sqrt{r^2 + 1}} \frac{w}{\| w \|}\right) - p' \right\|} = \frac{\sqrt{r'^2 - \frac{r'^2}{r^2 + 1}}}{\frac{r'}{\sqrt{r^2 + 1}}} = r,$$

so that $y \in V$, and the first case follows.

Note that the first case implies that $\Pi_{S'}(V_i - p + p') = H^+(w, w^+; S') \cup H^-(w, w^-; S')$.

In the second or third case σ cannot equal zero. We begin by assuming that $\sigma > 0$.

In the second case $p' \in V^-$. Since $p' \in V^-$, then $L^+(u, p')$ intersects V^+, if and only if either $u = \lambda(p - p')$ for some $\lambda > 0$, in which case $L^+(u, p') \cap V = L^+(u, p')$, or $u \in V_i^+ - p$, in which case there is a one-point intersection. Since the circle contained in V which contains p' has a tangent direction at p' which is orthogonal to the plane generated by w and $p - p'$, and since $w^{\times}$ is in this plane, and ${}^t w^{\times}(p - p') = 0$, then $P = P(w^{\times}, p')$ is the tangent plane to V at p'. If we note that ${}^t w^{\times}(-w) = \| w \|^2 \| p - p' \|^2 - ({}^t w(p - p'))^2 > 0$, then it follows that if ${}^t w^{\times}(u) < 0$, $L^+(u, p') \cap V^- = \varnothing$, while if ${}^t w^{\times}(u) > 0$, then there exists an $\varepsilon > 0$ such that $\{p' + tu : t \in (0, \varepsilon)\} \subseteq V_i^-$, so that $L^+(u, p')$ intersects V^- once, unless $u \in \bar{V}_i^- - p$, in which case there is no intersection. In P itself clearly only $L^+(\pm(p - p'), p')$ contribute to the projection. If we note that P separates V^- from V^+, this completes the proof for the second case with $\sigma > 0$.

In the third case, because $p' \in V_i^-$, $L^+(u, p')$ intersects $\bar{V}^-$ once, unless $u \in \bar{V}_i^- - p$, in which case there is no intersection, and $L^+(u, p')$ intersects $\bar{V}^+$ only if $u \in V_i^+ - p$, in which case there is a one-point intersection. This implies the result for the third case with $\sigma > 0$.

If $\sigma < 0$, then applying the already proven sections to $-w$ and rewriting the results in terms of w yields the proof for the second and third cases when $\sigma < 0$.

Now we turn to the fourth case. For each line $L = L(v, p)$ in V with ${}^t vw > 0$, let P be the plane containing L and p'. It is easy to see that $\Pi_{S'}(L)$ is an open great half circle which contains q^+ and starts at $p' + r'v/\| v \|$ and ends at $p' - r'v/\| v \|$. Thus, each line in V is mapped to an open half great circle containing q^+ and starting at a point y^+ on $(V^+ - p + p') \cap S' = \mathcal{C}^+$ and ending at the antipodal point $y^- = p' - (y^+ - p') \in (V^- - p + p') \cap S' = \mathcal{C}^-$. Conversely, each open half great circle containing q^+ and starting at a point $y^+ \in \mathcal{C}^+$ and ending at the antipodal point $y^- \in \mathcal{C}^-$ is the image of a unique line $L(y^+ - p', p)$ in V. Lemma 4 implies that only two great circles through q^+ are tangent to $\mathcal{C}^+$, one which we denote as $G^+ = G(n_+, q^+; S')$ with point of tangency p_+, and the other which we denote as $G^- = G(n_-, q^+; S')$ with point of tangency p_-^+. It also implies that G^+ and G^- are the only two great circles tangent to $\mathcal{C}^-$ with G^+ tangent at p_-^- and G^- tangent at p_+^-, and that $p_+^+ \in H^+(n_-, q^+; S')$, and $p_-^+ \in H^+(n_+, q^+; S')$.

Let $\mathcal{R}^+$ be the region bounded by G^+ and G^- whose closure contains $\mathcal{C}^+$, and $\mathcal{R}^-$ be the region bounded by G^+ and G^- whose closure contains $\mathcal{C}^-$; that is, $\mathcal{R}^+ = H^+(n_+, q^+; S') \cap H^+(n_-, q^+; S')$, and $\mathcal{R}^- = H^-(n_+, q^+; S') \cap H^-(n_-, q^+; S')$. Each great circle through q^+ contained in $\mathcal{R}^+ \cup \mathcal{R}^- \cup q^{\pm}$ intersects both $\mathcal{C}^+$ and $\mathcal{C}^-$ twice, and hence transversely, and no other great circles through q^+ except G^+ and G^- can intersect either $\mathcal{C}^+$ or $\mathcal{C}^-$. Each of these great circles determines two open half great circles through q^+ that begin at a point of $\mathcal{C}^+$. The two agree between $\mathcal{C}^+$ and $\mathcal{C}^-$, with one intersecting $\mathcal{C}^+$ but not intersecting $\mathcal{C}^-$, while the other intersects $\mathcal{C}^-$ but not $\mathcal{C}^+$. This implies that $\Pi_{S'}$ restricted to V is twofold only on that part of its image

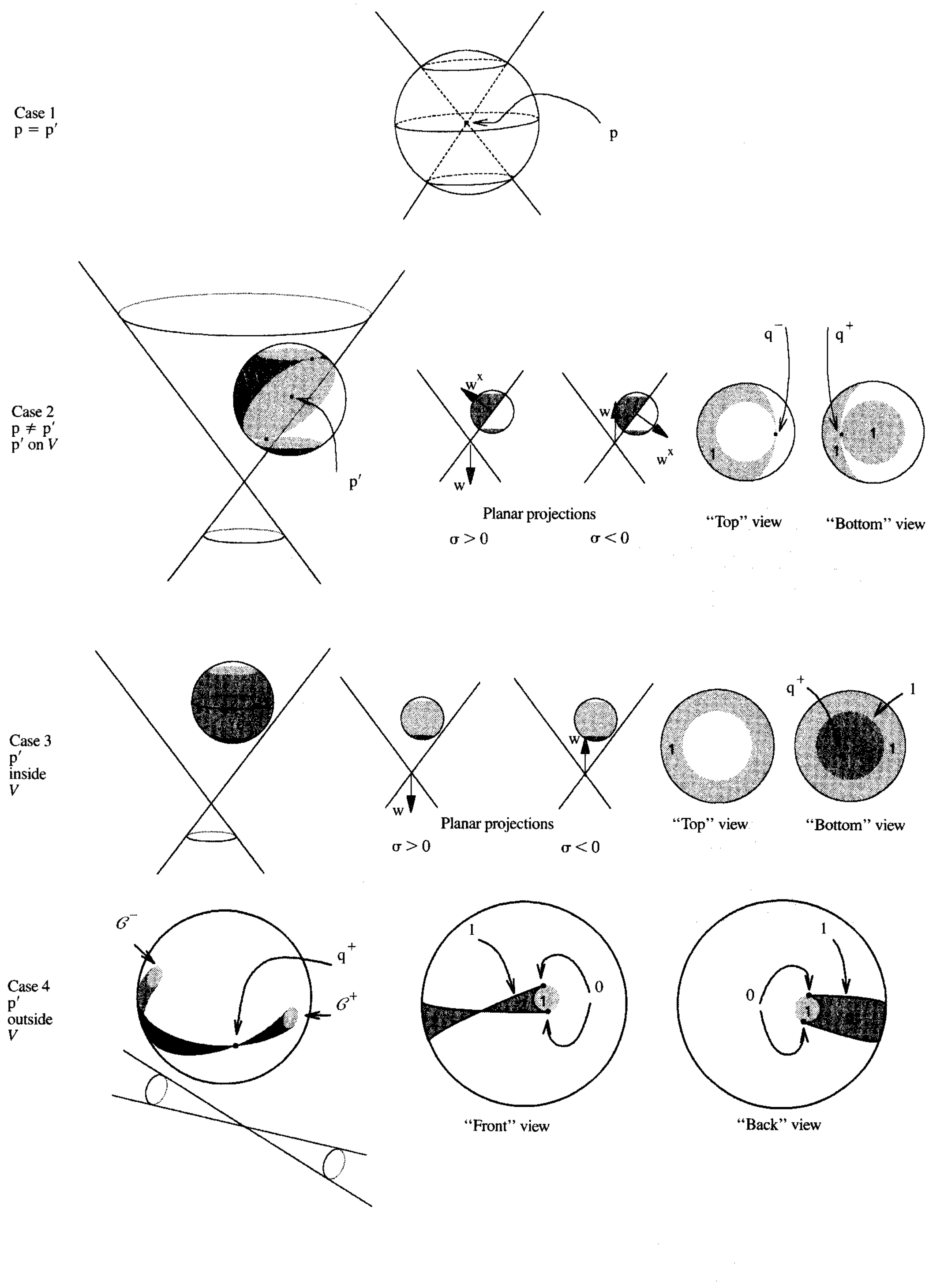


Figure 9

Projection of a cone onto a sphere.

obtained as the union of the open arcs between $\mathcal{C}^+$ and $\mathcal{C}^-$ of these open half circles, and is onefold elsewhere.

Let G be a great circle contained in $\overline{\mathcal{R}}^+ \cup \overline{\mathcal{R}}^-$. Let $\mathcal{B} = G(\mathrm{k}, \mathrm{p}'; S')$ be the circle through p_+^+, p_-^+, p_+^-, and p_-^-, with $\mathrm{q}^+ \in H^+(\mathrm{k}, \mathrm{p}'; S')$. $\mathcal{B}$ intersects $\mathcal{C}^+$ twice transversely at p_+^+ and p_-^+, and $\mathcal{B}$ intersects $\mathcal{C}^-$ twice transversely at p_-^+ and p_-^-. Hence

$$G \cap \Pi_{S'}(V) = (G \cap H^+(\mathrm{k}, \mathrm{p}'; S')) \cup (G \cap H^+(\mathrm{w}, \mathrm{w}^+; S')) \cup (G \cap H^-(\mathrm{w}, \mathrm{w}^-; S')).$$

Taking the union over all great circles $G \subseteq \overline{\mathcal{R}}^+ \cup \overline{\mathcal{R}}^-$ yields

$$\Pi_{S'}(V) = (\overline{\mathcal{R}}^+ \cap H^+(\mathrm{k}, \mathrm{p}'; S')) \cup H^+(\mathrm{w}, \mathrm{w}^+; S') \cup (\overline{\mathcal{R}}^- \cap H^+(\mathrm{k}, \mathrm{p}'; S') \cup H^-(\mathrm{w}, \mathrm{w}^-; S'),$$

as claimed.

All that remains to be done is to apply the formulas of Lemmas 4 and 5 to obtain explicit values for $\mathrm{n}_\pm$ and k.

In the terms of Lemma 4, q, p, v, and r here are q^+, p', $(r'/\sqrt{r^2+1})\mathrm{w}/\|\mathrm{w}\|$, and r'. Substituting these values for the formula for $\mathrm{n}_\pm$ yields

$$\mathrm{n}_\pm = \mp(\mathrm{q}^+ - \mathrm{p}') \times \frac{r'}{\sqrt{r^2+1}} \frac{\mathrm{w}}{\|\mathrm{w}\|} + \frac{\sqrt{\dfrac{r'^2}{r^2+1}\left(r'^2 - \dfrac{r'^2}{r^2+1}\right)}\left((\mathrm{q}^+ - \mathrm{p}') \times \dfrac{r'}{\sqrt{r^2-1}} \dfrac{\mathrm{w}}{\|\mathrm{w}\|}\right) \times (\mathrm{q}^+ - \mathrm{p}')}{\sqrt{r'^2\left(\dfrac{r'^4}{(r^2+1)^2} - \dfrac{r'^2}{r^2+1} \dfrac{1}{\|\mathrm{w}\|^2} ({}^t\mathrm{w}(\mathrm{q}^+ - \mathrm{p}'))^2\right)}}$$

$$= \frac{r'}{\sqrt{r^2+1}\,\|\mathrm{w}\|}\left[\mp\left(r' \frac{\mathrm{p}-\mathrm{p}'}{\|\mathrm{p}-\mathrm{p}'\|}\right) \times \mathrm{w} + \frac{\sqrt{r'^2 - \dfrac{r'^2}{r^2+1}}\left(\left(r' \dfrac{\mathrm{p}-\mathrm{p}'}{\|\mathrm{p}-\mathrm{p}'\|}\right) \times \mathrm{w}\right) \times \left(r' \dfrac{\mathrm{p}-\mathrm{p}'}{\|\mathrm{p}-\mathrm{p}'\|}\right)}{\sqrt{\dfrac{r'^4}{r^2+1} - r'^2 \dfrac{1}{\|\mathrm{w}\|^2}\left({}^t\mathrm{w}\left(r' \dfrac{\mathrm{p}-\mathrm{p}'}{\|\mathrm{p}-\mathrm{p}'\|}\right)\right)^2}}\right]$$

$$= \frac{r'^2}{\sqrt{r^2-1}\,\|\mathrm{w}\|\,\|\mathrm{p}-\mathrm{p}'\|}\left[\mp(\mathrm{p}-\mathrm{p}') \times \mathrm{w} \right.$$

$$\left. + \frac{\sqrt{r'^2\left(1 - \dfrac{1}{r^2+1}\right)}\,((\mathrm{p}-\mathrm{p}') \times \mathrm{w}) \times \left(r' \dfrac{\mathrm{p}-\mathrm{p}'}{\|\mathrm{p}-\mathrm{p}'\|}\right)}{\sqrt{\dfrac{r'^4}{(r^2+1)\|\mathrm{w}\|^2\|\mathrm{p}-\mathrm{p}'\|^2}\,(\|\mathrm{w}\|^2\|\mathrm{p}-\mathrm{p}'\|^2 - (r^2+1)({}^t\mathrm{w}(\mathrm{p}-\mathrm{p}'))^2)}}\right]$$

$$= \frac{r'^2}{\sqrt{r^2+1}\,\|\mathrm{w}\|\,\|\mathrm{p}-\mathrm{p}'\|}\left[\mp(\mathrm{p}-\mathrm{p}') \times \mathrm{w} \right.$$

$$\left. + \sqrt{\frac{\|\mathrm{w}\|^2 r^2}{\|\mathrm{w}\|^2\|\mathrm{p}-\mathrm{p}'\|^2 - (r^2+1)({}^t\mathrm{w}(\mathrm{p}-\mathrm{p}'))^2}}\,((\mathrm{p}-\mathrm{p}') \times \mathrm{w}) \times (\mathrm{p}-\mathrm{p}')\right],$$

which verifies the formula for $\mathrm{n}_\pm$.

By Lemma 5,

$$k = -\frac{r'}{\sqrt{r^2+1}} \frac{{}^t w}{\|w\|}\left(r' \frac{p-p'}{\|p-p'\|}\right) r'^2 \frac{r'}{\sqrt{r^2+1}} \frac{w}{\|w\|} + \frac{r'^4}{(r^2+1)^2} r' \frac{p-p'}{\|p-p'\|}$$

$$= \frac{r'^5}{(r^2+1)\|p-p'\|}\left[-\frac{1}{\|w\|^2}{}^t w(p-p')w + \frac{1}{r^2+1}(p-p')\right]$$

$$= \frac{r'^5}{(r^2+1)^2\|p-p'\|\|w\|^2}[-(r^2+1){}^t w(p-p')w + \|w\|^2(p-p')],$$

which verifies the formula for k and completes the proof. □

5. Intersections of cylinders and cones

In this section we consider the problem of obtaining the intersection of an arbitrary cylinder and an arbitrary cone. By the results of Section 1 for any fixed cylinder and fixed cone, it is sufficient to produce a base curve segmentation of a circle serving as the base curve of the cylinder or as the base curve of the cone. We have shown, moreover, that by using the partitioned projections of Sections 3 and 4 we can completely solve the problem, if we assume the existence of a two-dimensional modeling system capable of representing objects on a plane or a sphere that are bounded by circles, lines, and points and capable of intersecting such objects with a circle. The data describing the partitioned projections are not of great complexity, but do generally involve radicals. The internal intersections may involve more radicals. Thus, our assumption concerning the two-dimensional modeler implies the ability to manipulate radical expressions and also to evaluate their signs. In this section we show that at least in this case we need not make the assumption pertaining to the modeler or the implicit assumption regarding the ability to handle irrational quantities. Instead, we make a more common and less expansive assumption: the ability to perform arbitrary-precision rational calculation.

Let $V = V(w, p, r)$ and $C = C(v, q, r')$, and let $P = P(v, p)$, the plane orthogonal to C containing the vertex of V. We now sketch how we will proceed in the remainder of this section. First, we obtain $\Pi'_P(V)$, a partitioned projection of V onto P. Next we note that $C \cap P$ is a circle $\mathcal{C} = \mathcal{C}(c, r'; P)$, where $c = L(v, q) \cap P$, so that we may use $\mathcal{C}$ as the base curve of C that will be partitioned into a base curve segmentation by $\Pi'_P(V)$. In each of the three cases of Proposition 4, $\Pi'_P(V)$ partitions P into special points, segments of lines, and regions determined by the segments and points. $\mathcal{C} \cap \Pi_P(V)$ can be decomposed into finitely many, maximal, connected subsets, each one of which is contained in one of the subsets in $\Pi'_P(V)$. We denote the partitioning by $\mathcal{C} \cap \Pi'_P(V)$. Determining $\mathcal{C} \cap \Pi'_P(V)$ reduces to considering the ways that an arbitrary circle can interact with the partitioning. In fact, once we have determined the topological type of the intersection of $\mathcal{C}$ and $\Pi'_P(V)$, we are able to give explicit formulas for the base curve segmentation. To determine the specific topological configuration of $\mathcal{C} \cap \Pi'_P(V)$, we find all the possible configurations, and by a careful analysis of them can define polynomials in the parameters of C and V whose signs determine the topological type of $\mathcal{C} \cap \Pi'_P(V)$. We define cases in terms of these signs. Thus, by testing the signs of polynomials we perform a case analysis, and then use the formulas appropriate to that case to define a base curve segmentation. If we provide a parameterization of each arc of the base curve, then by determining the parameter value type of the domain of the arc and appealing to formulas Γ_1 through Γ_4 of Section 1, we obtain $C \cap V$. As with any case-by-case analysis, the difficulty is in determining and organizing the large number of subcases. In this regard we are aided by the fact that we are defining the cases in terms of signs of polynomials. Thus, there is a natural notation, and if we ensure that all possible sign combinations are considered, we can be sure that no situations can escape scrutiny.

To begin, if we define

$$f_1 : (x_1, x_2, s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \rightarrow \|x_2\|^2 \|x_1\|^2 - (s^2+1)({}^t x_2 x_1)^2,$$

then by Section 2, $v + p$ is inside, on, or outside V depending upon whether $f_1(w, v, r)$ is less than, equal to, or greater than zero. Thus, by Proposition 4, $\Pi_P(V)$ doubly covers $P\backslash p$ and singly covers p, if $f_1(w, v, r) < 0$; $\Pi_P(V)$ singly covers $P\backslash L(w \times v, p)$ and maps the line $L(v, p)$ onto p, if $f_1(w, v, r) = 0$; and $\Pi_P(V)$ doubly covers the interior of

$$V = \{p + \alpha u_+ + \beta u_- : \alpha, \beta \in \mathbb{R}, \alpha\beta \geq 0\},$$

where

$$u_\pm = (\|w \times v\|^2 - r^2({}^t vw)^2)^{1/2} v \times (w \times v) \pm r\|v\|^2\|w\| w \times v$$

and singly covers $L(u_\pm, p)$, if $f_1(w, v, r) > 0$. We refer to these cases as the (−) case, the (0) case, and the (+) case, respectively.

Since P is perpendicular to C, $P \cap C$ is a circle in P of radius r' whose center, c, is on the axis of C, L(v, q). A trivial calculation yields

$$c = L(v, q) \cap P = q + \frac{{}^t v(p - q)}{\| v \|^2} v.$$

Thus $P \cap C = \mathcal{C} = \mathcal{C}(c, r'; P)$.

• *The simple cases*

The (−) case is particularly simple. Here $\Pi'_p(V)$ partitions P into the point, p, and the rest of P. The circle, $\mathcal{C}$, in P has the point inside it, on it, or outside it, which is equivalent to $\| c - p \|^2 - r'^2$ being negative, zero, or positive. Since

$$\begin{aligned} \| c - p \|^2 &= \| q + {}^t v(p - q) \| v \|^{-2} v - p \|^2 \\ &= (\| q - p \|^2 \| v \|^2 - ({}^t v(q - p))^2) \| v \|^{-2} \\ &= \| (q - p) \times v \|^2 \| v \|^{-2}, \end{aligned}$$

if we define

$$f_2 : (x_1, x_2, x_3, s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \\ \rightarrow \| (x_1 - x_2) \times x_3 \|^2 - s^2 \| x_3 \|^2,$$

it follows that $f_2(q, p, v, r)$ negative, zero, or positive implies that p is respectively inside, on, or outside $\mathcal{C}$. We refer to these subcases as the (− −) case, the (−0) case, and the (− +) case. In the (−0) case, $\Pi'_p(V)$ decomposes $\mathcal{C}$ into p and $\gamma(p, p; p)$, while in the other two cases $\Pi'_p(V) \cap \mathcal{C} = \mathcal{C}$. The arcs are doubly covered and p is simply covered.

The (0) case is almost as simple. $\Pi'_p(V)$ partitions P into the point, p, the two half-lines of $L^\times = L(w \times v, p)$ separated by p, and the two half-planes with boundary $L^\times$. We partition $\mathcal{C}$ by its intersections with $L^\times$. Thus each arc is singly covered, while if $p \in \mathcal{C} \cap \Pi_p(V)$, it is covered by a line of C. If $L^\times$ intersects $\mathcal{C}$ in two points, they can be found explicitly by solving for s in $\| s w \times v + p - c \|^2 = r'^2$, and we denote them as $p_{\times\pm} = p + {}^t(w \times v)(c - p) \pm (r'^2 \| w \times v \|^2 - \| (w \times v) \times (c - p) \|^2)^{1/2} \| w \times v \|^{-2} w \times v$. If $p \in \mathcal{C}$, then the root in this expression reduces to $\pm\, {}^t(w \times v)(c - p)$, so that $\{p_{\times-}, p_{\times+}\} = \{p, p_{\times *}\}$, where $p_{\times *} = p + 2\,{}^t(w \times v)(c - p) \| w \times v \|^{-2} w \times v$. When $L^\times$ is tangent to $\mathcal{C}$, the root vanishes, and we find that $p_{\times T}$, the point of tangency, satisfies $p_{\times T} = p + {}^t(w \times v)(c - p) \| w \times v \|^{-2} w \times v$. Finally, let $p^\perp = c + r' v \times (w \times v) / \| v \times (w \times v) \|$. If $L^\times$ is tangent to $\mathcal{C}$, then $p^\perp = p_{\times T}$, but otherwise $p^\perp$ does not belong to $L^\times$.

We may use f_2 again to determine three subcases (0−), (00), and (0+) depending upon whether p is inside, on, or outside $\mathcal{C}$. For the (0−) case, $L^\times$ must partition $\mathcal{C}$ into two arcs, so that $\mathcal{C} \cap \Pi'_p(V) = \gamma(p_{\times+}, p_{\times-}; p^\perp) \cup \gamma^c(p_{\times+}, p_{\times-}; p^\perp)$. For the (00) case, either $\mathcal{C}$ is tangent to $L^\times$, or $L^\times$ partitions $\mathcal{C}$ into two arcs. This tangency is equivalent to c being r' distant from $L^\times$, and since $v \times (w \times v)$ lies in P and is orthogonal to $L^\times$, it is equivalent to $| {}^t(\| v \times (w \times v) \|^{-1} v \times (w \times v))(c - p) |$ equaling r', which in turn is equivalent to $| {}^t(\| v \times (w \times v) \|^{-1} v \times (w \times v))(q - p) |$ equaling r', since $v \times (w \times v)$ is also orthogonal to v. Thus, if we define

$$f_3 : (x_1, x_2, x_3, x_4, s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \\ \rightarrow ({}^t(x_1 \times (x_2 \times x_1))(x_3 - x_4))^2 \\ - s^2 \| x_1 \times (x_2 \times x_1) \|^2,$$

$\mathcal{C}$ is tangent to $L^\times$ only if $f_3(v, w, q, p, r') = 0$. Because $p \in \mathcal{C}$, $f_3(v, w, q, p, r') \le 0$, so it is reasonable to denote the cases of tangency and nontangency as (000) and (00−). Now, in the (00−) case $\mathcal{C} \cap \Pi'_p(V) = p \cup \gamma(p, p_{\times *}; p^\perp) \cup \gamma^c(p, p_{\times *}; p^\perp)$, while in the (000) case $\mathcal{C} \cap \Pi'_p(V) = p \cup \gamma^c(p, p; p)$.

In the (0+) case either $\mathcal{C}$ intersects $L^\times$ in two points, so that $\Pi'_p(V) \cap \mathcal{C}$ is the union of the two arcs $\gamma(p_{\times+}, p_{\times-}; p^\perp)$ and $\gamma^c(p_{\times+}, p_{\times-}; p^\perp)$; or $\mathcal{C}$ is tangent to $L^\times$ at $p_{\times T}$, so that $\Pi'_p(V) \cap \mathcal{C} = \gamma^c(p_{\times T}, p_{\times T}; p_{\times T})$; or $\mathcal{C}$ is disjoint from $L^\times$, so that $\mathcal{C} \cap \Pi'_p(V) = \mathcal{C}$, depending upon whether the distance from c to $L^\times$ is less than, equal to, or greater than r'. As in the (00) case, this is equivalent to the sign of $f_3(v, w, q, p, r')$ being negative, zero, or positive, and we designate the cases as (0+−), (0+0), and (0++).

Figure 10 depicts examples of the nine cases considered thus far.

• *Cases where $\Pi_p(V)$ is a planar cone*

The (+) case is more complex, or at least more complicated. Here $\Pi_p(V)$ is the planar cone, V, and so we denote $\Pi'_p(V)$ as V′. Let us denote $L(u_\pm, p)$ by $L_\pm$. We partition $\mathcal{C}$ by its intersections with $L_\pm$. Thus, each arc in $\mathcal{C} \cap V'$ is doubly covered, and its boundary points are singly covered, since V is doubly covered on its interior and singly covered on its boundary. As above, when $L_\pm$ intersects twice, let $p_{\pm\pm} = p + ({}^t u_\pm(c - p) \pm (r'^2 \| u_\pm \|^2 - \| u_\pm \times (c - p) \|^2)^{1/2}) \| u_\pm \|^{-2} u_\pm$, and when $p \in \mathcal{C}$, let $p_{\pm *} = p + 2\,{}^t u_\pm(c - p) \| u_\pm \|^{-2} u_\pm$. If $L_\pm$ is tangent to $\mathcal{C}$, let $p_{\pm T} = p + {}^t u_\pm(c - p) \| u_\pm \|^{-2} u_\pm$.

As in the (−) and (0) cases, we begin by considering the relationship of p and $\mathcal{C}$. As before, p is inside, on, or outside $\mathcal{C}$ depending on the sign of $f_2(q, p, v, r')$, and we designate the corresponding subcases as (+−), (+0), and (++).

In the (+−) case p is inside $\mathcal{C}$, and so L_+ and L_- must each intersect $\mathcal{C}$ in two points, $p_{+\pm}$ and $p_{-\pm}$, respectively. Thus, $V' \cap \mathcal{C}$ is the union of p_{++}, p_{-+} and the arc connecting them, $\gamma^c(p_{++}, p_{-+}; p_{+-})$, and p_{+-}, p_{--} and the arc connecting them, $\gamma^c(p_{+-}, p_{--}; p_{++})$.

In the (+0) case, $p \in \mathcal{C}$ and so L_+ intersects $\mathcal{C}$ at the second point, p_{+*}, unless it is tangent to $\mathcal{C}$ at p, and L_- intersects $\mathcal{C}$ at the second point, p_{-*}, unless it is tangent to $\mathcal{C}$ at p. Since $V = V(u_+, u_-, p)$, $z \in \mathbb{R}^3$ belongs to V if and only if

$$-\alpha \le \frac{{}^t(w \times v)(z - p)}{{}^t(v \times (w \times v))(z - p)} \le \alpha$$

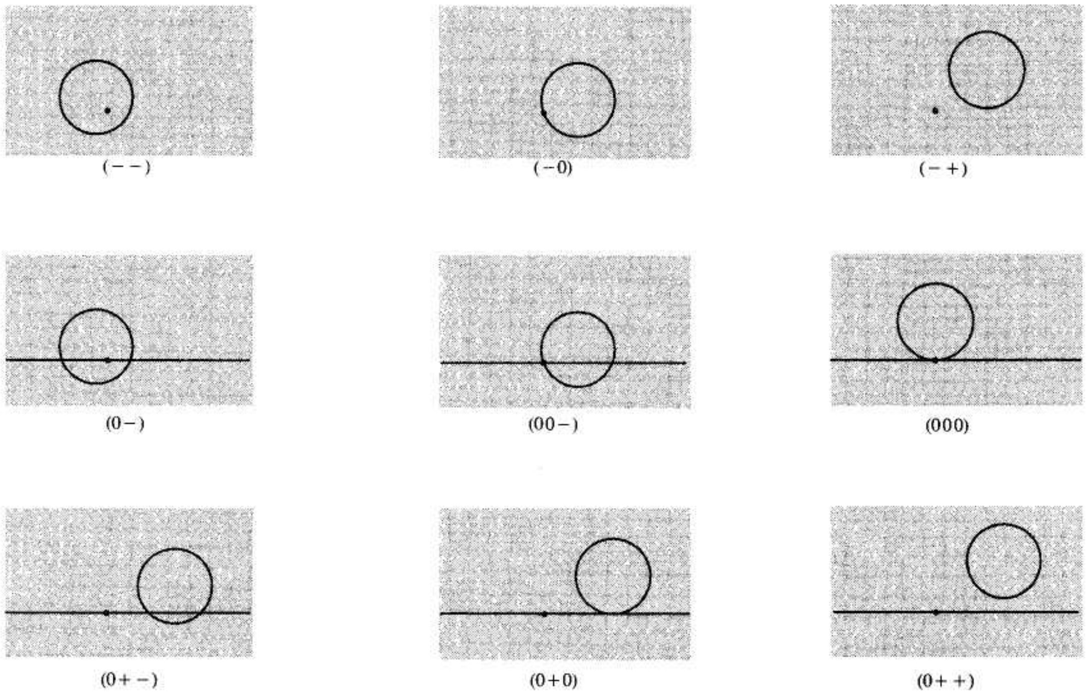


Figure 10

Examples of the nine subcases of the (−) and (0) cases.

for $\alpha = r\| w \|(\| w \times v \|^2 - r^2({}^t vw)^2)^{-1/2}$. We need to classify the tangent to $\mathcal{C}$ at p, which equals $(p - c) \times v = (p - q) \times v$, and so we define

$$f_4 : (x_1, x_2, x_3, s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$$

$$\rightarrow ({}^t(x_1 \times x_2)x_3)^2(\| x_1 \times x_2 \|^2 - s^2({}^t x_1 x_2)^2)$$

$$- s^2 \| x_1 \|^2({}^t(x_2 \times (x_1 \times x_2))x_3)^2.$$

Now, the tangent to $\mathcal{C}$ at p is in the interior of V, or is in the boundary of V (and hence L_+ or L_- is tangent to $\mathcal{C}$ at p), or is exterior to V, depending upon whether $f_4(w, v, (p - q) \times v, r)$ is negative, zero, or positive, and we designate these cases as $(+0-)$, $(+00)$, and $(+0+)$.

Since the tangent to $\mathcal{C}$ at p is in the interior of V in the $(+0-)$ case, it follows that $\mathcal{C} \cap V'$ is composed of p, p_{+*}, p_{-*}, $\gamma^c(p, p_{+*}; p_{-*})$, the arc connecting p and p_{+*}, and $\gamma^c(p, p_{-*}; p_{+*})$, the arc connecting p and p_{-*}.

In the $(+00)$ case, $p - q$ is perpendicular to either L_+ or L_-. Assume that $p - q$ is perpendicular to L_+. One sees easily that $\mathcal{C} \cap V$ must be an arc of $\mathcal{C}$ connecting p and p_{-*}. Since $L(v \times (w \times v), p) \subset V$, $L(v \times (w \times v), p) \cap \mathcal{C} \subset \mathcal{C} \cap V$. Obviously, p is one point of the intersection, and since L_+ is tangent to $\mathcal{C}$ at p, there is a second point, $p_{vwv*} = p + 2{}^t(v \times (w \times v))(c - p)\| v \times (w \times v)\|^{-2} v \times (w \times v)$, in $L(v \times (w \times v), p) \cap \mathcal{C}$. Thus, $\mathcal{C} \cap V' = p \cup p_{-*} \cup \gamma(p, p_{-*}; p_{vwv*})$. Similarly, if $p - q$ is perpendicular to L_-, then $\mathcal{C} \cap V' = p \cup p_{+*} \cup \gamma(p, p_{+*}; p_{vwv*})$.

When we consider the $(+0+)$ case, the tangent to $\mathcal{C}$ at p is exterior to V, so p is isolated in $\mathcal{C} \cap V$. Thus $\mathcal{C} \cap V'$ is composed of p, p_{+*}, p_{-*}, and the arc of $\mathcal{C}$, $\gamma^c(p_{+*}, p_{-*}; p)$.

Figure 11 depicts examples of the $(+-)$ and $(+0)$ subcases.

Intersections of $\mathcal{C}$ with $L_\pm$

In the remaining subcases of the (+) case, p is outside $\mathcal{C}$. If c is at a distance greater than r' from L_+ and L_-, then $\mathcal{C} \cap V$ is either empty or $\mathcal{C}$ itself. If c is exactly r' from L_+ (or L_-), then L_+ (or L_-) is tangential to $\mathcal{C}$. If the distance to L_+ from c is less than r', then L_+ divides $\mathcal{C}$ into two arcs, only one of which can be contained in $\mathcal{C} \cap V$. Let

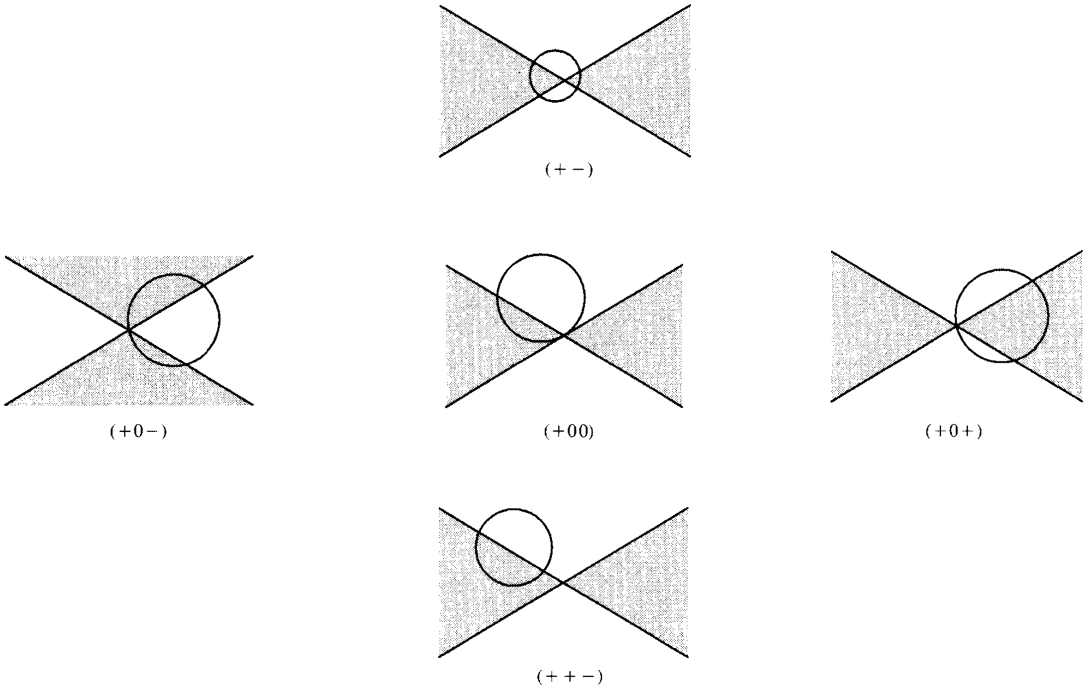


Figure 11

Examples of the (+ −), (+ 0), and (+ + −) subcases.

$$u_{\pm}^{\perp} = r\| v \| \| w \| v \times (w \times v)$$

$$\mp \| v \| (\| w \times v \|^2 - r^2({}^t vw)^2)^{1/2} w \times v.$$

$u_{+}^{\perp}$ is perpendicular to u_{+}, and $u_{-}^{\perp}$ is perpendicular to u_{-}. ${}^t u_{+}^{\perp} u_{-} > 0$, and ${}^t u_{-}^{\perp} u_{+} > 0$.

$$\| u_{+} \|^2$$

$$= \| u_{-} \|^2 = \| u_{+}^{\perp} \|^2 = \| u_{-}^{\perp} \|^2$$

$$= (r^2 \| v \|^2 \| w \|^2 + (\| w \times v \|^2 - r^2({}^t vw)^2)) \| v \|^2 \| w \times v \|^2$$

$$= (\| w \times v \|^2 + r^2(\| v \|^2 \| w \|^2 - ({}^t vw)^2)) \| v \|^2 \| w \times v \|^2$$

$$= (1 + r^2) \| w \times v \|^4 \| v \|^2.$$

${}^t u_{+}^{\perp}(c - p)/\| u_{+}^{\perp} \|$ is the signed distance from c to L_{+}, so that $({}^t u_{+}^{\perp}(c - p) - r' \| u_{+}^{\perp} \|)({}^t u_{+}^{\perp}(c - p) + r' \| u_{+}^{\perp} \|) = ({}^t u_{+}^{\perp}(c - p))^2 - r'^2 \| u_{+}^{\perp} \|^2$ is negative, zero, or positive when $\mathcal{C}$ intersects L_{+} twice, L_{+} is tangent to $\mathcal{C}$, or $\mathcal{C}$ is disjoint from L_{+}, respectively. Similarly, $({}^t u_{-}^{\perp}(c - p))^2 - r'^2 \| u_{-}^{\perp} \|^2$ is negative, zero, or positive when $\mathcal{C}$ intersects L_{-} in two points, L_{-} is tangent to $\mathcal{C}$, or $\mathcal{C}$ and L_{-} are disjoint. From

$${}^t u_{+}^{\perp}(c - p) - r' \| u_{+}^{\perp} \|$$

$$= \| v \| (r \| w \| {}^t(v \times (w \times v))(c - p)$$

$$- (\| w \times v \|^2 - r^2({}^t vw)^2)^{1/2}\, {}^t(w \times v)(c - p)$$

$$- r' \| w \times v \|^2 (1 + r^2)^{1/2}),$$

we see that ${}^t u_{+}^{\perp} - r' \| u_{+}^{\perp} \|$ is not a polynomial in its parameters. If we define $\alpha = r \| w \| {}^t(v \times (w \times v))(c - p)$, $\beta = (\| w \times v \|^2 - r^2({}^t vw)^2)^{1/2}\, {}^t(w \times v)(c - p)$, and $\gamma = r' \| w \times v \|^2 (1 + r^2)^{1/2}$, then ${}^t u_{+}^{\perp}(c - p) - r' \| u_{+}^{\perp} \| = \| v \| (\alpha - \beta - \gamma)$, ${}^t u_{+}^{\perp}(c - p) + r' \| u_{+}^{\perp} \| = \| v \| (\alpha - \beta + \gamma)$, ${}^t u_{-}^{\perp}(c - p) - r' \| u_{-}^{\perp} \| = \| v \| (\alpha + \beta - \gamma)$, and ${}^t u_{-}^{\perp}(c - p) + r' \| u_{-}^{\perp} \| = \| v \| (\alpha + \beta + \gamma)$. Thus, none of these expressions are polynomials in their parameters. Moreover, since $(\alpha - \beta - \gamma)(\alpha - \beta + \gamma) = \alpha^2 - 2\alpha\beta + \beta^2 - \gamma^2$, and $\alpha\beta$ is not polynomial, the products above are not polynomials. However, since

$(\alpha - \beta - \gamma)(\alpha - \beta + \gamma)(\alpha + \beta - \gamma)(\alpha + \beta + \gamma) = (\alpha^2 - \beta^2 - \gamma^2)^2 - 4\beta^2\gamma^2$, the product of these four expressions is always a polynomial. Thus, we define

$$
\begin{aligned}
f_5 &: (x_1, x_2, x_3, x_4, s_1, s_2) \\
&\in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \\
&\rightarrow (s_1^2 \| x_1 \|^2 ({}^t(x_2 \times (x_1 \times x_2))(x_3 - x_4))^2 \\
&\quad - (\| x_1 \times x_2 \|^2 - s_1^2({}^t x_1 x_2)^2)({}^t(x_1 \times x_2)(x_3 - x_4))^2 \\
&\quad - s_2^2 \| x_1 \times x_2 \|^4 (1 + s_1^2))^2 - 4(\| x_1 \times x_2 \|^2 \\
&\quad - s_1^2({}^t x_1 x_2)^2)({}^t(x_1 \times x_2)(x_3 - x_4))^2 s_2^2 \| x_1 \times x_2 \|^4 (1 + s_1^2),
\end{aligned}
$$

and distinguish three subcases $(++-)$, $(++0)$, and $(+++)$, where $f_5(w, v, q, p, r, r')$ is negative, zero, or positive.

In the $(++-)$ case, one and only one of $({}^t u_+^\perp(c - p))^2 - r'^2 \| u_+^\perp \|^2$ and $({}^t u_-^\perp(c - p))^2 - r'^2 \| u_-^\perp \|^2$ is negative and corresponds to the one and only one of L_+ and L_- that intersects $\mathcal{C}$ in two points, with $\mathcal{C}$ being disjoint from the other. Since L_+ and L_- are symmetric about $L(w \times v, p)$ and $L(v \times (w \times v), p)$, the quadrant defined by $L(w \times v, p)$ and $L(v \times (w \times v), p)$ which contains c determines which of L_+ and L_- is closer to c. Thus, if $({}^t(w \times v)(c - p))({}^t(v \times (w \times v))(c - p)) > 0$, c is closer to L_+, and so L_+ intersects $\mathcal{C}$ at p_{+-} and p_{++}, and $\mathcal{C} \cap V$ is an arc between them. One easily sees that $c + \sigma r' u_+^\perp / \| u_+^\perp \| \in \mathcal{C} \cap V$, where σ is the sign of ${}^t(v \times (w \times v))(c - p)$, so that $\mathcal{C} \cap V' = p_{+-} \cup p_{++} \cup \gamma(p_{+-}, p_{++}; c + \sigma r' u_+^\perp / \| u_+^\perp \|)$. Similarly, if $({}^t(w \times v)(c - p))({}^t(v \times (w \times v))(c - p)) < 0$, then $\mathcal{C} \cap V' = p_{--} \cup p_{-+} \cup \gamma(p_{--}, p_{-+}; c + \sigma r' u_-^\perp / \| u_-^\perp \|)$. We note that the expression cannot equal zero, because this implies that c is on $L(w \times v, p)$ or $L(v \times (w \times v), p)$, and symmetry would imply that $\mathcal{C}$ intersects both L_+ and L_-, or neither of them. See Figure 11 for an example of the $(++-)$ case.

The dual cone and the relative sizes of cones

Let the dual cone of V, denoted V^*, be defined as $V^* = V(u_+^\perp, u_-^\perp, p)$. V^* is of interest, because it is composed of those points $z \in P$ such that z is closer to $L^+(u_+, p)$ (hereafter denoted L_+^+) than to $L^-(u_+, p)$ (hereafter L_+^-), if and only if z is closer to $L^+(u_-, p)$ (hereafter L_-^+) than to $L^-(u_-, p)$ (hereafter L_-^-). This implies that any circle whose center is in V^* can interact only with L_+^+ and L_-^+, or with L_+^- and L_-^-, and that any circle whose center is in V^{*c} interacts only with L_+^+ and L_-^-, or with L_+^- and L_-^+. Any circle with p exterior to it whose center is on the boundary of V^* can interact with at most one of L_+ and L_-. In a manner similar to the $(+-)$ case, we find that $z \in \mathbb{R}^3$ belongs to V^* if and only if

$$
-\alpha \leq \frac{{}^t(w \times v)(z - p)}{{}^t(v \times (w \times v))(z - p)} \leq \alpha
$$

for $\alpha = (\| w \times v \|^2 - r^2({}^t vw)^2)^{1/2}(r \| w \| \| v \|^2)^{-1}$, and so we define

$$
\begin{aligned}
f_6 &: (x_1, x_2, x_3, x_4, s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \\
&\rightarrow ({}^t(x_1 \times x_2)(x_3 - x_4))^2 s^2 \| x_1 \|^2 \| x_2 \|^4 \\
&\quad - (({}^t x_2 \times (x_1 \times x_2))(x_3 - x_4))^2 (\| x_1 \times x_2 \|^2 - s^2({}^t(x_1 x_2)),
\end{aligned}
$$

and distinguish three subcases of the $(++0)$ case which we denote as $(++0-)$, $(++00)$, and $(++0+)$ depending upon the sign of $f_6(w, v, q, p, r)$, which in turn depends upon whether c belongs to the interior of V^*, to the boundary of V^*, or to the interior of V^{*c}.

In the $(++0-)$ case, if $c \in V$ and $\mathcal{C}$ is sufficiently "small" relative to V, $\mathcal{C}$ must be contained in V and have a single tangency to ∂V. As $\mathcal{C}$ grows "larger," it may acquire a second tangency, or even cease to be contained in V. In order to quantify these vague ideas of smaller and larger, we recall that by Lemma 1 there is a unique planar cone, $V(z, z', p)$, that circumscribes $\mathcal{C}$, and we denote this cone as $V(\mathcal{C})$. The Pythagorean theorem implies that the square of the tangent of the half angle between $L(z, p)$ and $L(z', p)$ equals $r'^2(\| p - c \|^2 - r'^2)^{-1}$, where

$$
\begin{aligned}
\| p - c \|^2 &= \| q + {}^t v(p - q)v / \| v \|^2 - p \|^2 \\
&= \| \| v \|^2 (q - p) - {}^t v(q - p)v \|^2 \| v \|^{-4} \\
&= (\| v \|^2 \| q - p \|^2 - ({}^t(v(q - p))^2) \| v \|^{-2} \\
&= \| v \|^{-2} \| v \times (q - p) \|^2.
\end{aligned}
$$

If $\mathcal{C}$ intersects both L_+^+ and L_-^+ or intersects both L_+^- and L_-^-, the half angle of $V(\mathcal{C})$ must be greater than or equal to that of V, which is equivalent to $r^2 \| w \|^2 \| v \|^2 (\| w \times v \|^2 - r^2({}^t vw)^2)^{-1} \leq r'^2 \| v \|^2 (\| q - p) \times v \|^2 - r'^2 \| v \|^2)^{-1}$, since the tangent function is increasing. Thus, if we define

$$
\begin{aligned}
f_7 &: (x_1, x_2, x_3, x_4, s_1, s_2) \\
&\in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \\
&\rightarrow s_1^2 \| x_1 \|^2 (\| x_2 \times (x_3 - x_4) \|^2 - s_2^2 \| x_2 \|^2) \\
&\quad - s_2^2(\| x_1 \times x_2 \|^2 - s_1^2({}^t x_1 x_2)^2),
\end{aligned}
$$

$f_7(w, v, q, p, r, r')$ is negative, zero, or positive when V has an angle smaller than, equal to, or greater than $V(\mathcal{C})$. Similarly, if

$$
\begin{aligned}
f_8 &: (x_1, x_2, x_3, x_4, s_1, s_2) \\
&\in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \\
&\rightarrow (\| x_2 \times (x_3 - x_4) \|^2 - s_2^2 \| x_2 \|^2)(\| x_1 \times x_2 \|^2 \\
&\quad - s_1^2({}^t x_1 x_2)^2) - s_1^2 s_2^2 \| x_1 \|^2 \| x_2 \|^4,
\end{aligned}
$$

$f_8(w, v, q, p, r, r')$ is negative, zero, or positive when V^c has an angle smaller than, equal to, or greater than $V(\mathcal{C})$.

In the $(++0-)$ case we define three subcases $(++0--)$, $(++0-0)$, and $(++0-+)$, when p is inside V, on ∂V, or exterior to V, which is implied by the sign of $f_4(w, v, q - p, r)$. We partition the $(++0--)$ case into three cases $(++0---)$, $(++0--0)$, and $(++0--+)$ by using the sign of $f_7(w, v, q, p, r, r')$, that is, depending on the relative sizes of V and $V(\mathcal{C})$.

In the $(++0---)$ case, since $\mathcal{C}$ is tangential to ∂V, $c \in V^*$, $c \in V$, and V is smaller than $V(\mathcal{C})$, it follows that $\mathcal{C}$ intersects ∂V at one point tangentially and at two points transversely. The three points either belong to L_+^+ and L_-^+ or belong to L_+^- and L_-^- with the two points of transverse intersection on one half-line and the point of tangential intersection on the other. Thus $\mathcal{C} \cap V$ is composed of the arc that connects the two points of transverse intersection and contains the point of tangential intersection. As before, if $({}^t(w \times v)(c - p))({}^t(v \times (w \times v))(c - p) > 0$, then c is closer to L_+ than L_-, and so L_+ intersects $\mathcal{C}$ in the two points p_{+-} and p_{++}, so that $\mathcal{C} \cap V' = p_{+-} \cup p_{++} \cup p_{-T} \cup \gamma^c(p_{+-}, p_{-T}; p_{++}) \cup \gamma^c(p_{++}, p_{-T}; p_{+-})$. If, on the other hand, $({}^t(w \times v)(c - p))({}^t(v \times (w \times v))(c - p)) < 0$, then $\mathcal{C} \cap V' = p_{--} \cup p_{-+} \cup p_{+T} \cup \gamma^c(p_{--}, p_{+T}; p_{-+}) \cup \gamma^c(p_{-+}, p_{+T}; p_{--})$. Symmetry again shows that the expression cannot take the value zero.

In the $(++0--0)$ case, since $\mathcal{C}$ is tangent to ∂V, $V(\mathcal{C})$ and V share a common boundary. In fact, since $c \in V$ and the two cones are of equal size, they must coincide. Thus, $\mathcal{C}$ is contained in V and meets ∂V tangentially at two points, one on L_+ and the other on L_-. It follows from symmetry that $c \in L(v \times (w \times v), p)$ so that $c_{vwv\pm} = c \pm r'v \times (w \times v)/\|v \times (w \times v)\| \in \mathcal{C} \cap V$. Thus we may write $\mathcal{C} \cap V' = p_{-T} \cup p_{+T} \cup \gamma(p_{-T}, p_{+T}; c_{vwv-}) \cup \gamma(p_{-T}, p_{+T}; c_{vwv+})$.

In the $(++0--+)$ case, $\mathcal{C}$ is tangential to ∂V, $c \in V$, and $V(\mathcal{C})$ is smaller than V, so that V contains $\mathcal{C}$ and ∂V intersects $\mathcal{C}$ in only the one tangential point. Once again, if $({}^t(w \times v)(c - p))({}^t(v \times (w \times v))(c - p)) > 0$, then c is closer to L_+, so that $\mathcal{C} \cap V' = p_{+T} \cup \gamma^c(p_{+T}, p_{+T}; p_{+T})$, while if $({}^t(w \times v)(c - p))({}^t(v \times (w \times v))(c - p)) < 0$, $\mathcal{C} \cap V' = p_{-T} \cup \gamma^c(p_{-T}, p_{-T}; p_{-T})$.

$c \in \partial V$ in the $(++0-0)$ case; that is, $c \in L_+$ or L_-, and so $\mathcal{C}$ intersects this line transversely in two points. $\mathcal{C}$ must also intersect ∂V tangentially, and because $c \in V^*$, all three of these points must lie either in L_+^+ and L_-^+ or in L_+^- and L_-^-. This implies that the $(++0-0)$ case is equivalent to the $(++0---)$ case, which we have already considered.

c is exterior to V in the $(++0-+)$ case. Since $c \in V^*$, $\mathcal{C}$ intersects only L_+^+ and L_-^+, or it intersects only L_+^- and L_-^-. Thus, either $\mathcal{C}$ is exterior to V, except at the point of tangential intersection, or $\mathcal{C}$ intersects one of L_+ and L_- twice transversely and is tangential to the other. In the latter case $\mathcal{C}$ is forced to intersect $L(v \times (w \times v), p)$. $\mathcal{C}$ intersects this line if and only if the distance from c to it is less than or equal to r', or $({}^t(w \times v/\|w \times v\|)(c - p))^2 - r'^2 \le 0$. Accordingly, we define

$$f_9 : (x_1, x_2, x_3, x_4, s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$$

$$\rightarrow ({}^t(x_1 \times x_2)(x_3 - x_4))^2 - s^2\|x_1 \times x_2\|^2,$$

and define subcases $(++0-+-)$ and $(++0-++)$ based on whether $f_9(w, v, q, p, r)$ is negative or positive. Case $(++0-+-)$ is equivalent to case $(++0---)$. In the $(++0-++)$ case, $\mathcal{C} \cap V$ is a single point which equals p_{+T} if $({}^t(w \times v)(c - p))({}^t(v \times (w \times v))(c - p))$ is positive, and equals p_{-T} if it is negative. If f_9 were to take the value zero, $\mathcal{C}$ would be tangential to $L(v \times (w \times v), p)$, a bisector of L_+ and L_-, and thus could be tangential to neither.

The analysis of the $(++0-)$ subcases is now completed. Next, we consider the $(++00)$ case. Since c is on the boundary of V^*, $\mathcal{C}$ can intersect only one of L_+ and L_-. Since $\mathcal{C}$ is tangential to ∂V, $\mathcal{C}$ must intersect ∂V at a single point. If c is interior to V, then $\mathcal{C} \subset V$, and so $\mathcal{C} \cap V'$ is as in the $(++0--+)$ case, while if c is exterior to V, then $\mathcal{C} \subset V^c$, and $\mathcal{C} \cap V'$ is as in the $(++0-++)$ case. The sign of $f_4(w, v, q - p, r)$ distinguishes these cases: the $(++00-)$ case for the interior circle, where $f_4 < 0$, and the $(++00+)$ case for the exterior circle, where $f_4 > 0$. If f_4 were to have the value of zero, then $c \in \partial V$, and thus would imply that $\mathcal{C}$ intersects ∂V in at least two points, contradicting $c \in \partial V^*$, and $\mathcal{C}$ intersects ∂V tangentially.

The analysis of the $(++0+)$ subcases is very similar to that of the $(++0-)$ subcases. In fact, if we replace V with V^c and make the concomitant changes, the partitioning into subcases is identical. First, we define three subcases $(++0+-)$, $(++0+0)$, and $(++0++)$, depending upon whether c is interior to V^c, in ∂V^c, or exterior to V^c, which is equivalent to $-f_4(w, v, q - p, r)$ being negative, zero, or positive. In the $(++0+-)$ case, we further partition into $(++0+--)$, $(++0+-0)$, and $(++0+-+)$ when the size of V^c is less than, equal to, or greater than the size of $V(\mathcal{C})$, that is, by the sign of $f_8(w, v, q, p, r, r')$. In the $(++0++)$ case we partition by the sign of $f_9(v, w \times v, p, q, r)$, which implies that the distance from c to $L(w \times v, p)$ is less than, equal to, or greater than r', which in turn implies whether $\mathcal{C}$ intersects $L(w \times v, p)$ transversely, tangentially, or disjointly.

We could now analyze these subcases in the manner of the $(++0-)$ case, but rather we omit the details and summarize the results. The $(++0+--)$ case is analogous to the $(++0---)$ case. Here we find that $c \in V^{*c}$, $c \in V^c$, and V^c smaller than $V(\mathcal{C})$ imply that $\mathcal{C} \cap \partial V$ has one point of tangential intersection and two points of transverse intersection and that $\mathcal{C} \cap V' = p_{-T} \cup \gamma^c(p_{+-}, p_{++}; p_{-T})$, if $({}^t(w \times v)(c - p))({}^t(v \times (w \times v))(c - p)) > 0$, while $\mathcal{C} \cap V' = p_{+T} \cup \gamma^c(p_{--}, p_{-+}; p_{+T})$, otherwise. The $(++0+-0)$ case is analogous to the $(++0--0)$ case. In this case, we again find two points of tangential intersection, so that $\mathcal{C} \cap V = p_{+T} \cup p_{-T}$. The remaining subcases are equivalent to previously considered cases. In particular, the $(++0+0)$ case and the $(++0++-)$ case are equivalent to the $(++0+--)$ case, the $(++0+-+)$ case is equivalent to the $(++0-++)$ case, and the $(++0+++)$ case is equivalent to the $(++0--+)$ case.

Figure 12 contains a pictorial representation of the partitioning scheme of the $(++0)$ cases and examples of the six distinguishable types of base curve partitioning.

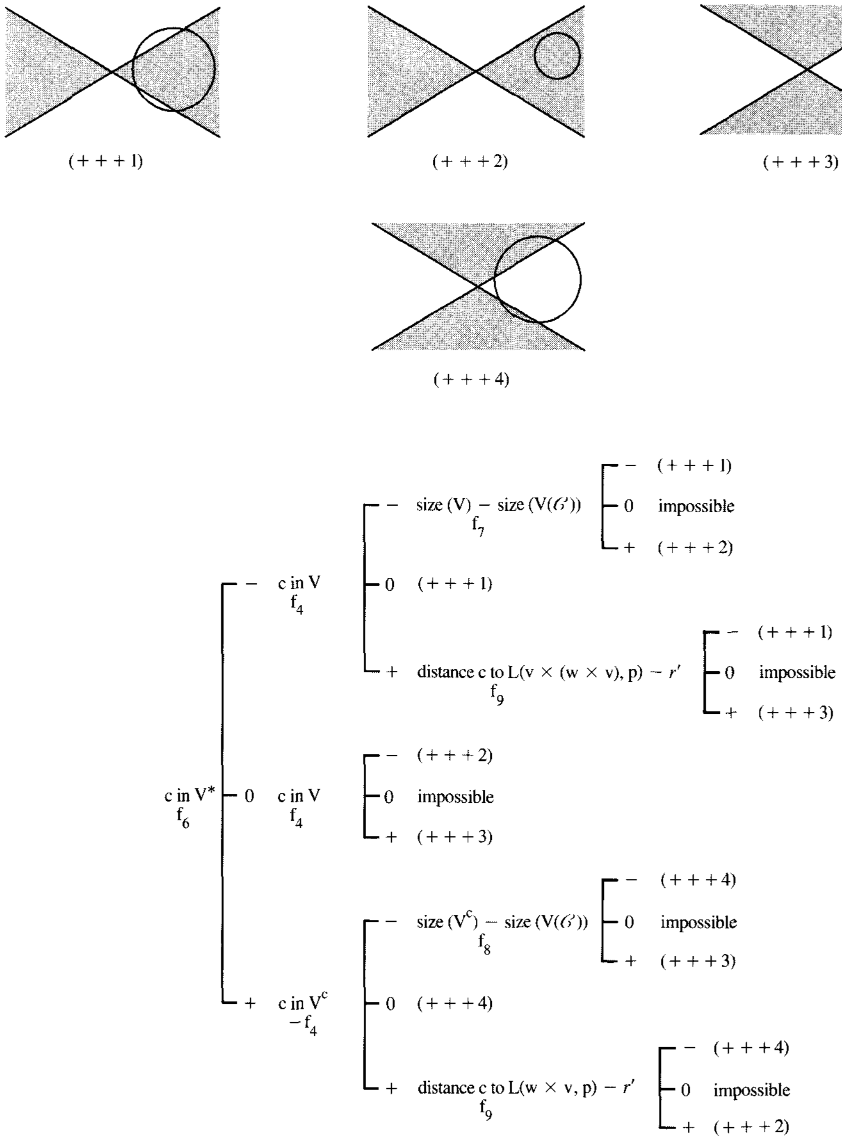


Figure 12

Partitioning scheme for the (+ + 0) cases and examples of the four distinguishable cases.

The final cases

Only the (+ + +) case remains to be analyzed. In this case $f_5(w, v, q, p, r, r')$ is positive, so that both $({}^t u_+^\perp (c - p))^2 - r'^2 \| u_+^\perp \|^2$ and $({}^t u_-^\perp (c - p))^2 - r'^2 \| u_-^\perp \|^2$ are negative or both are positive; that is, either $\mathcal{C}$ transversely intersects each of L_+ and L_- or $\mathcal{C}$ is disjoint from each of them. To discover the ways in which this may happen, we follow the scheme developed for the (+ + 0) case. First, we define three subcases dependent on the sign of $f_6(w, v, q, p, r)$, which is equivalent to c being interior to V^*, in ∂V^*, or interior to V^{*c}. We partition each of these subcases by the sign of $f_4(w, v, q - p, r)$ to distinguish when c is interior to V, in ∂V, or exterior to V. In the (+ + + − −) case we use the sign of $f_7(w, v, q, p, r, r')$ to partition further into subcases that correspond to V being smaller than, equal to, or greater than $V(\mathcal{C})$. Similarly, the $(++++-+)$ case is partitioned by the sign of $f_8(w, v, q, p, r, r')$ into subcases depending on the relative sizes of V^c and $V(\mathcal{C})$. In the $(+++-+)$ case we define three subcases depending upon whether $L(v \times (w \times v), p)$ transversely intersects $\mathcal{C}$, tangentially intersects $\mathcal{C}$, or is disjoint from $\mathcal{C}$, and distinguish the three by the sign of $f_9(w, v, q, p, r)$. Finally, the sign of $f_9(v, w \times v, q, p, r)$ partitions the $(+++++)$ case into subcases where $L(w \times v, p)$ intersects $\mathcal{C}$ transversely, tangentially, or disjointly.

In the $(+++-0)$ case, $c \in \partial V$, so that $\mathcal{C}$ must intersect one of, and hence both of, L_+ and L_-. Since $c \in V^*$, it follows that $\mathcal{C} \cap V' = p_{-+} \cup p_{++} \cup \gamma^c(p_{-+}, p_{++}; p_{--}) \cup p_{--} \cup p_{+-} \cup \gamma^c(p_{--}, p_{+-}; p_{++})$.

In the $(+++0)$ case, $c \in \partial V^*$, and so can intersect only one of, and hence neither of, L_+ and L_-.

In the $(+++0-)$ case, $c \in V$, so $\mathcal{C} \cap V' = \mathcal{C}$, while in the $(+++0+)$ case $c \in V^c$, so that $\mathcal{C} \cap V$ is empty.

In the $(++++0)$ case, $c \in \partial V$, so that $\mathcal{C}$ must intersect one of, and hence both of, L_+ and L_-. Since $c \in V^{*c}$, it follows that $\mathcal{C} \cap V' = p_{--} \cup p_{-+} \cup \gamma^c(p_{--}, p_{-+}; p_{++}) \cup p_{+-} \cup p_{++} \cup \gamma^c(p_{+-}, p_{++}; p_{--})$.

Similar analysis reveals the following equivalences. The $(+++---)$ case and the $(+++-+-)$ case are equivalent to the $(+++-0)$ case. The $(++++--+)$ case and the $(++++++)$ case are equivalent to the $(+++0-)$ case. The $(+++-++)$ case and the $(++++-+)$ case are equivalent to the $(+++0+)$ case. Finally, the $(++++--)$ case and the $(+++++-)$ case are equivalent to the $(++++0)$ case. Since the remaining cases that we have defined can easily be shown to lead to contradictory assumptions, we have completed the analysis of the $(+++)$ case, and hence of 24 possible topological configurations of $\mathcal{C} \cap \Pi'_P(V)$.

Figure 13 contains a pictorial representation of the partitioning scheme of the $(+++)$ cases and examples of the six distinguishable types of base curve segmentation.

• *Determination of parameter value type*

In the previous section we obtained a base curve segmentation for each of the possible configurations of $\mathcal{C} \cap \Pi'_P(V)$. It is now easy to choose a parameterization of each point and arc of the base curve segmentation to obtain a domain segmentation, and we will assume this has been done. By the results of Section 1, once we have determined the parameter value type of each subset of the domain segmentation, we can apply the one of the four maps, Γ_1 to Γ_4, which is appropriate to each subset, and so explicitly obtain $C \cap V$. As noted in Section 1, only in type 2–3 subsets of the base curve segmentation, that is, only in subsets singly covered by $\Pi_P(V)$, can there be ambiguity, with it being possible for such a subset to correspond to a subset of the domain segmentation of parameter value type 2 or 3, and hence to lead to Γ_2 or Γ_3. We showed that if for any t_0 in the subset $A(t_0) = 0$, the parameter value type of the subset is 2 at all values, and hence we apply Γ_2, while if $A(t_0) \neq 0$, the parameter value type of the subset is 3 at all values, and hence we must apply Γ_3. In fact, this test is unnecessary, because this information is already implicitly contained in the decomposition into cases, as we now show.

As we have seen, by Section 2 for any $x \in \mathbb{R}^3$, $x \in V$, if and only if $f(x) = \|x - p\|^2 \|w\|^2 - (r^2 + 1)({}^t w(x - p))^2 = 0$, and since f is a quadratic polynomial in the coordinates of x, it serves to define V as a quadric surface. If we define matrix Q as $Q = \|w\|^2 I - (r^2 + 1) w {}^t w$ for the identity I, then it is easy to see that ${}^t x Q x$ is the pure quadratic part of f, and so for this choice of defining equation for V we find by Section 1 that $A(t) = {}^t v Q v = \|v\|^2 \|w\|^2 - (r^2 + 1)({}^t w v)^2$, which we recognize to be $f_1(w, v, r)$. By [2], any other defining equation, $\tilde{f}$, leads to an $\tilde{A}(t)$ such that $\tilde{A}(t) = sA(t)$ for some nonzero scalar s. Thus, if γ is a type 2–3 subset in the base curve segmentation and g its corresponding subset in the domain segmentation, $t \in g \to \Gamma_2(t)$ is the pre-image of γ in $C \cap V$, if $\mathcal{C} \cap \Pi'_P(V)$ is in one of the (0) subcases, while $t \in g \to \Gamma_3(t)$ is the pre-image of γ in $C \cap V$, if $\mathcal{C} \cap \Pi'_P(V)$ is in one of the $(-)$ or $(+)$ subcases.

• *A comment on stability*

Case-by-case analysis tends to be laborious. Ours is no exception. However, the partitioning we have defined in our analysis provides an unexpected benefit of stability. Let $\mathbb{R}^3_*$ be $\mathbb{R}^3$ minus the origin, and $\mathbb{R}_+$ be the positive real numbers. $P = \mathbb{R}^3_* \times \mathbb{R}^3 \times \mathbb{R}_+$ can serve as a parameter space of cylinders or cones, so that in a natural way we can identify the topological product, $P_i = P \times P$, with the parameter space of intersection problems of cylinders and cones. Formally, we have used the polynomials f_1 through f_9 to partition P_i into subsets where an intersection problem can easily be solved by fixed formulas. Let $\mathcal{S}$ be one of these subsets and $z = (v, q, r', w, p, r)$ be an element of $\mathcal{S}$. If we follow the methods discussed above, then z determines $f(x; z)$, the defining equation of the cone; a projection, $\Pi(z)$; and a base curve circle, $\mathcal{C}(z)$, and hence a base curve segmentation, $\gamma(z)$. Each of these objects changes smoothly as z moves through $\mathcal{S}$, as can easily be seen from the formulas we have developed for them. In particular, since $\mathcal{C}(z)$ changes smoothly, it is possible to choose a parameterization of each $\mathcal{C}(z)$, so that z smoothly determines a domain segmentation, $g(z) = \{g_i(z)\}$, and even a smooth set-valued map from z to the intersection, $\Gamma: z \to \{\Gamma(g_i; z, i)\}$, where $\Gamma(g_i; z, i)$ denotes the image of g_i under the one of the maps $\Gamma_1(z)$ through $\Gamma_4(z)$ which is appropriate for the i-th subset of g(z) and is invariant over $\mathcal{S}$ by construction. Thus, in a sense, rather than solving the specific problem of $C \cap V$, we have simultaneously solved all such problems in a manner such that when we restrict considerations to any subset of the partition of P_i, the inherent instability of the intersection problems, as

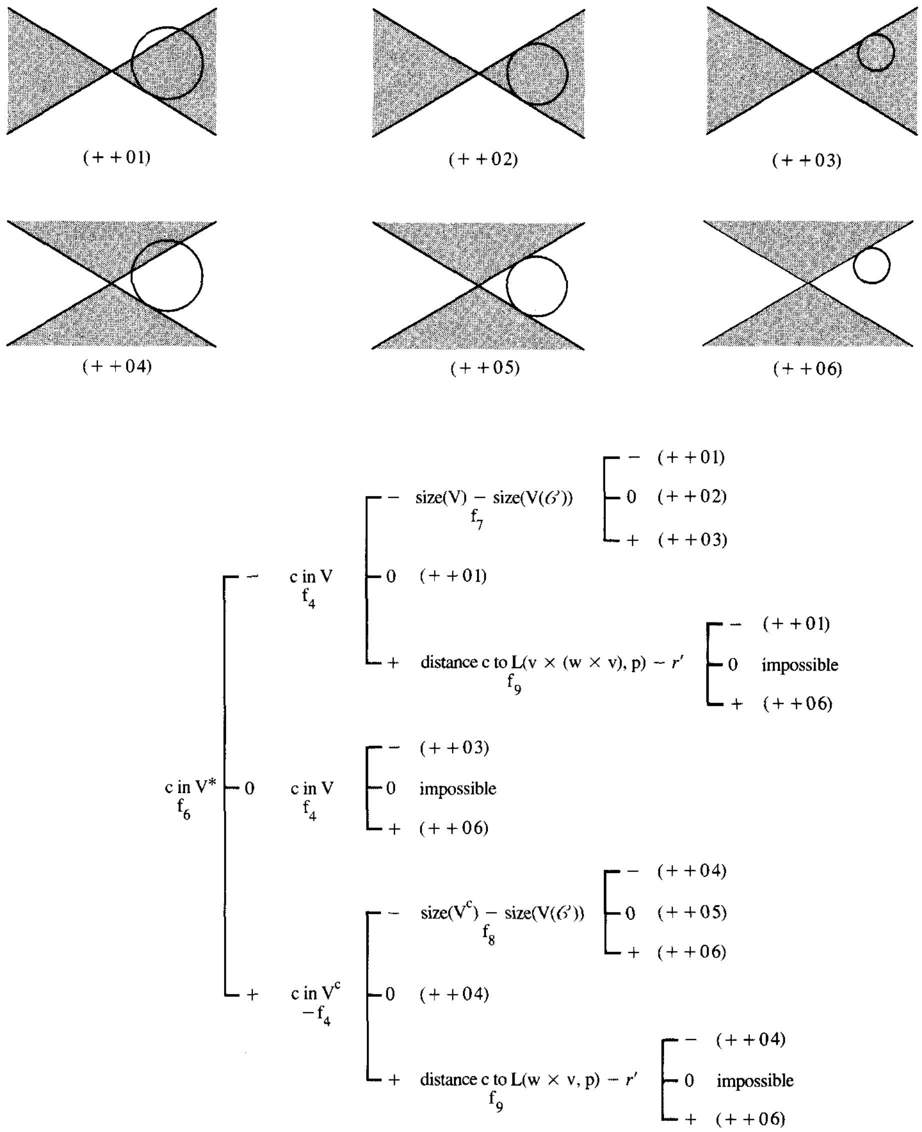


Figure 13

Partitioning scheme for the (+ + +) cases and examples of the six distinguishable cases.

exemplified by the two-cylinder intersection considered in the Introduction, is replaced by the smooth map Γ whose values are all topologically equivalent.

6. Concuding remarks

In this paper we have treated the problem of computing the intersection of two natural quadric surfaces. In Section 1 we began by considering in detail a restriction of Levin's method of intersecting two quadric surfaces to the case where both surfaces are natural quadrics and one is a ruled quadric, that is, either a cone or a cylinder. We saw that a solution for the purposes of geometrical modeling would be obtained by this method, if we could also produce a domain segmentation, but that the normal means of producing a domain segmentation led to the necessity to solve fourth-degree polynomials. However, by using explicit knowledge of the partitioned projections of natural quadrics, we found that a related object, a base curve segmentation, was computable using only the intersection between lines and circles, and evaluations of polynomials. Section 1 closed with a proof that a base curve segmentation led directly to a domain segmentation, so that we could conclude that the exact intersection of two natural quadrics could be produced by a two-dimensional modeling system supporting objects bounded by circles, lines, and points on a sphere or plane. In Sections 3 and 4 we presented the projections of natural quadrics onto planes and spheres, and thus offered a complete description of the objects necessary for implementation of the method of Section 1. In Section 5 we observed that the two-dimensional modeler assumed in Section 1 implied the ability to handle data involving irrationalities exactly. Since this is in fact a strong assumption, we chose to weaken it by assuming only the ability to perform exact rational arithmetic. The techniques of Section 1 motivated the search for a partition of the parameter space of intersections between cylinders and cones into subsets where a base curve segmentation could easily be explicitly obtained. In this setting, the partition was defined by the signs of polynomials in the parameters of the problem. The section closed by noting that when consideration is restricted to one of the subsets of this partition, the instability usually associated with intersection problems is replaced by a smooth map from the parameters of the intersection problem to the intersection itself.

At its most fundamental level, this paper yields a reduction in the difficulty of finding a domain segmentation for use in intersecting two natural quadrics from the computation of roots of a fourth-degree equation to second-degree geometric calculations. It is reasonable to ask if such a simplification can be extended to the natural domain of Levin's method: the quadric surfaces. If we consider the intersection of two parallel elliptic cylinders, then projecting and intersecting to obtain a base curve yields a final problem of intersecting two ellipses, which in general leads to a fourth-degree polynomial equation. Although we have only shown that domain segmentation by the usual means leads to fourth-degree polynomial equations for natural quadrics, it is also possible to show that this is true for all quadrics. Thus, in this case, the general technique and using projections can lead to the same degree of difficulty, so that we must recognize that projections are not a panacea for the algebraic complexity of domain segmentation in all quadric surface intersection problems. However, the technique has a wider range of applicability than just the natural quadrics: similar reductions can be obtained for other types of quadric surface intersections using little more than the tools that have been developed here. For example, consider the problem of intersecting C, a general cylinder (elliptic, parabolic, or hyperbolic) with Γ, a general cone. If P is the plane containing the vertex of Γ and orthogonal to the axis of C, then by a nonisotropic change of scale we can transform Γ to a right circular cone, $\tilde{\Gamma}$. We can then find the partitioned projection of $\tilde{\Gamma}$ onto P using Proposition 4, and by inverting the scaling obtain the partitioned projection of $\tilde{\Gamma}$ onto P, which can again be described by points and lines in P. The intersection of C with P is an ellipse, a parabola, or a hyperbola, which serves as a base curve for C. Thus, we can produce a base curve segmentation by intersecting lines and points with ellipses, parabolas, and hyperbolas. Each of these problems reduces to a simple second-degree polynomial calculation, and, as before, we can proceed to find a domain segmentation.

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References and notes

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5. Each of these parameterizations has a problem at a single point. The transcendental one would also map 2π to (1, 0), while the rational one maps no real number to (−1, 0). Both cause only minor implementation problems, and neither affects the following arguments, and so are ignored.
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