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AUTHOR: WILLIAM A. MARTIN

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A Step by Step Computer Solution of Three Problems in Non-numerical Analysis

This memo describes the step by step solution of three problems from different fields of applied mathematics. These problems are solved by typing a series of computer commands for the manipulation of symbolic mathematical expressions. These commands are best typed at the PDP-6 console, so that the Type 30 display and the wider range of keyboard symbols can be used. The syntax of commands typed at the PDP-6 will be described. These commands are translated into a string of symbols which are sent to CTSS, where they are parsed into a LISP expression, which is then evaluated.

The mathematical operators which are available in the system will be described and then the step by step solution of each of the problems will be given.

II. The Mathematical Operators

The commands typed at the PDP-6 are similar to Algol statements. The commands are typed and executed one at a time. More complex operations involving the definition and alteration of commands and the introduction of more pneumonics and man machine interaction will be described later. The commands consist of infix operators, functions, and variables. Functions and variables can be subscripted and any subexpression can be quoted. A sample command which requires most of the notation is:

```
#'E1←'(X + Y) * 'DRV(':T,1,DRV('U,2,E1)) + !E2,20 + '(F[I,J](X,Y))↑2#
```

In words: the name E1 is assigned to the expression which is the sum of three terms. The first term is the product of (X + Y) with the unevaluated first derivative with respect to lower case T of the second derivative with respect to U of the expression currently named E1. The second term is the 20th subexpression of a displayed expression currently named E2; this subexpression has been indicated with the light pen. The final term is the square of a subscripted function of X and Y. The notation may seem somewhat complex, but as will be seen, a complex notation is required to express in a compact way the many small steps required to solve a particular problem.

The infix operators are:

A←B	B is given name A. As such it is written on the disk. The value of ← is B.
!A,N	The Nth subexpression of A is the value of !. Intensify the desired subexpression of A by pointing to its main connective with the light pen. Then type !A, and the computer will type N. If the expression has no main connective, point to one of its arguments and type ;!A instead of !A. Consider all minus signs to be unary.
A=B	Equate A and B
A+B	A plus B
A−B	A minus B
A*B	A times B
A/B	A divided by B A/B*C is equivalent to A/(B*C)
A↑B	A to the power B

The functional, subscripting, and set notation is:

A(C,D,E)	A is a function with arguments C,D, and E.
A[I,J] (C,D,E)	$A_{I,J}$ is a function with arguments C,D,and E.
A[I,J]	$A_{I,J}$ is a variable
(A,B,C)	This is a set with three elements. By convention (A)= A

' Either an expression or a variable name can be quoted. A function name always stands for itself. Quoting a function name means that its arguments will be evaluated but that the function will not be evaluated. For example let F(X,Y) = X-Y be a function and let X and Y be names for A; then

F(X,Y) evaluates to 0

'F(X,Y) evaluates to F(A,A)

F('X,Y) evaluates to X-A

'(F(X,Y)) evaluates to F(X,Y).

Quoting a function or variable name does not quote its subscripts. Numbers are taken as quoted automatically.

: Causes the letters which follow it to be lower case for purposes of display.

As in CTSS, there are two editing characters:

? Deletes all the characters of a command back to the initial #.

" Deletes only the immediately preceding character.

"#" must be the first and last character of every command. ";" causes the current intensified subexpression to be raised one level. For example, if the A in $A^B + C$ is intensified, then when ; is typed A^B will be intensified.

Other available operators are:

ALLSUMEXPAND(EXP) — Applies SUMEXPAND to every summation in expression EXP.

BRINGOVER(EXP, X) — Subexpression X, which has been indicated with the light pen is brought to the other side of equation EXP.

COLLECT(EXP,SET) — Top level terms in EXP are collected on powers of the expressions in set SET.

DEPENDENCE(EXP) — Returns a set of the variable and function names in EXP.

DELSUBST(EXP,OLDDEL, NEWDEL) — $\frac{dx}{d\ \mathrm{OLDDEL}} \rightarrow \frac{dx}{d\ \mathrm{NEWDEL}}$ for each such subexpression in EXP

DRV[X_1,N_1, ..., X_n,N_n, Y) — Differentiate Y N_i times with respect to X_i, for each i.

DRVDO(EXP,X) — All indicated derivatives with respect to X in EXP are carried out as far as possible.

DRVFACTOR(EXP, X, N) — $\frac{d^{N+M}f}{dx^{N+M}} \rightarrow \frac{d^M}{dx^M} \left(\frac{d^N f}{dx^N} \right)$ for each such subexpression in EXP.

DRVZERO(EXP,X) — All derivatives with repect to X in EXP are set equal to zero.

EVALUATE(EXP,SET) — SET is a set of equations; whenever the left side of one of these equations can be matched to a subexpression in EXP, the right hand side is substituted. The left sides must be variables or functions. A match occurs whenever a binding of the function variables and subscripts can be made.

EXCHANGE(EXP) — If the top level connective of EXP is binary, its arguments are exchanged, right to left.

EXPAND(EXP) — Multiplies out all expressions of the form a*(b+c) in EXP. In addition,

$$\frac{d}{dx}(a+b) \rightarrow \frac{da}{dx} + \frac{db}{dx}$$

FACTOROUT(EXP,FACTOR,Y) — The factor FACTOR is factored from each term of EXP. The third argument Y is optional. If Y is present, the factor FACTOR is renamed Y.

GROUP(SET) — The set SET of terms which have been indicated by the light pen in EXP are grouped within the associated sum or product. The value of GROUP is the grouped set of terms.

LEFT(EXP) — Returns the left argument of the main binary connective of EXP.

LIMIT(EXP,X,N) — Determines the limiting value of EXP as X approaches N.

MULTIPLYTHROUGH(EXP,X) — Multiplies each top level term of EXP by X.

NEWNAME() — Creates a name of the form Fn, where n is an integer.

NORMPOLY(EXP,X) — Every sum in EXP is treated as a polynomial in X and a power of X is factored out so that the lowest power of X in the polynomial will be zero.

REPLACE(E,X,Y) — Expression X replaces Y in the expression named E. Y is a term indicated with the light pen or a group of

	terms indicated with GROUP. If the light pen has been used to construct X, the resulting expression position is named HOLE. HOLE can then be used for the third argument. If X is equal to NIL, then the third argument is omitted from the expression named E.
RIGHT(EXP)	Returns the right argument of the main binary connective of EXP.
SIMPLIFY(EXP)	Simplifies expression EXP.
SOLVE(EXP,X)	Solves equation EXP for variable X as far as possible.
SPLIT(EXP)	Subparts of EXP are named and replaced by their names in EXP, so that EXP will contain less than 100 subexpressions.
SUBSTITUTE(EXP,X,Y)	Substitute X for each occurrence of Y in EXP.
SUMEACH(EXP)	$\Sigma(a+b) \rightarrow \Sigma a + \Sigma b$
SUMEXPAND(EXP)	Expands the finite summation EXP.
TERM(EXP,N)	Returns the Nth argument of the top level connective of EXP, or NIL if there is no Nth argument.
TRUNCATE(EXP,VAR,N)	Expands EXP up to power N in variable VAR.
SUM(I,N1,N2,Y)	Sum expression Y for values of I from N1 to N2.
ITG(X,L1,L2,Y)	Integrate Y with respect to X between limits L1 and L2.

Expressions which are assigned names are kept on the disk. The expression most recently computed always has the name LAST. When A←B is executed, if A is not "LAST" and is already the name of an expression, then this old value of A is given the name OLD. Thus, if A←A+2 is executed and then is found to be incorrect, the old value of A can be retrieved.

Operators used for input-output and disk storage are:

EDISPLAY(E)	Displays the expression named E on the PDP-6 scope.
EPRINT(E)	Prints out the internal form of the expression named E with PLS, PRD, EQN, and PWR in infix form; the other operators in prefix form.
EDELETE(E)	Deletes expression named E from the disk.

This completes the description of the PDP-6 commands.

The Poincare-Lighthill Procedure
Applied to $\ddot{x} + \omega^2 x = \varepsilon x^3$

The Poincare-Lighthill procedure is typical of a number of procedures used to find the first few terms in the asymptotic expansion of the function which is the solution to a mildly non-linear differential equation. The equation chosen here is that for a harmonic oscillator with a small forcing function. These solution procedures involve assuming a series expansion in powers of the small parameter ε for one or more of the parameters and variables in the equation, substituting these series into the differential equation, and thus obtaining a series of relations between the coefficients of like powers of ε. Each of these equations is then treated in turn by whatever methods seem appropriate. Thus it is in general necessary to see these equations before the next steps in the solution process can be determined.

When a typed command has been completed, the machine makes a response of acknowledgment. This standard response will be omitted in the dialogue to follow; only the typed commands and the displayed equations will be shown. A running discussion of the dialogue is included and the displayed equations are shown on the pages following their use. These equations were plotted by the CALCOMP plotter. A photograph of these same equations is shown at the end of the section. The reader should be aware that in the equation syntax used, more than one line of an expression can occur over a divide bar or within brackets. This is illustrated by equation Q16 in the last section.

Enter the differential equation.

```
#'E1←'(DRV(:T,2,X(:T)) + OMEGA↑2*X(:T) = EP*(X(:T))↑3)#
#EDISPLAY('E1)#
```

A new independent variable τ is introduced in order to stretch the time. Type in expressions for series expansions for X and t in terms of functions of τ. That is, a solution for X(τ) rather than for X(t) will be found. Since t depends on τ, equation E1 can be used to find an equation in derivatives of X(τ). As a final step the inverse relationship τ(t) will be found, so that X(τ) will give X(t).

```
#'E2←'(X(TAU) = SUM(I,0,INF,EP↑I*X[I](TAU)))#
#EDISPLAY('E2)#

#'E3←'(:T(TAU) = TAU + SUM(J,1,INF,EP↑J*:T[J](TAU)))#
#EDISPLAY('E3)#
```

In order to substitute dτ for dt it is necessary to apply the transformation

$$\frac{d^2x}{dt^2} \rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

to equation E1.

```
#'E4←DRVFACTOR(E1,':T,1)#
```

(E1) $$\frac{d^2}{dt^2}X(t)+\omega^2\cdot X(t)=\epsilon\cdot X(t)^3$$

(E2) $$X(\tau)=\sum_{I=0}^{\infty}\epsilon^{I}\cdot X_{I}(\tau)$$

(E3) $$t(\tau)=\tau+\sum_{J=1}^{\infty}\epsilon^{J}\cdot t_{J}(\tau)$$

Display E4 for comparison with the substituted result below.

```
#EDISPLAY('E4)#
```

Now substitute $d\tau$ for dt, and $X(\tau)$ for $X(t)$

```
#'E5←SUBSTITUTE(DELSUBST(E4 ,'(DEL(:T)), '(DEL(TAU))/DRV('TAU,1,RIGHT(E3))),
                                                '(X(TAU)),'(X(:T)))#
```

```
#EDISPLAY('E5)#
```

Now substitute the series for $X(\tau)$ and perform the indicated differentiation with repect to τ.

```
#'E6←DRVDO(SUBSTITUTE(E5, RIGHT(E2),'(X(TAU))),'TAU)#
#EDISPLAY('E6)#
```

Now expand both sides to first order in ε.

```
#'E7←TRUNCATE(E6,'EP,1)#
```

The zero order terms form the harmonic oscillator equation; the solution can be written down by inspection as $A\cos \omega\tau$. Use the light pen to form an equation of the first order terms.

```
#'E8←!E7,6=;!E7,88#
```

```
#EDISPLAY('E8)#
```

Bring the terms in $X_1(\tau)$ to the left side of the first order equation. Substitute for $X_0(\tau)$ and carry out the indicated differentiation.

[illegible]

$$\frac{d}{dt}\left[\frac{d}{dt}X(t)\right]+\omega^{2}\cdot X(t)=\varepsilon\cdot X(t)^{3}$$

[illegible]

$$\frac{\left\{\dfrac{d}{d\tau}\dfrac{\left[\dfrac{d}{d\tau}X(\tau)\right]}{\left\{\sum_{J=1}^{\infty}\left[\dfrac{d}{d\tau}t_{J}(\tau)\right]\cdot\varepsilon^{J}+1\right\}}\right\}}{\left\{\sum_{J=1}^{\infty}\left[\dfrac{d}{d\tau}t_{J}(\tau)\right]\cdot\varepsilon^{J}+1\right\}}+\omega^{2}\cdot X(\tau)=\varepsilon\cdot X(\tau)^{3}$$

[illegible]

$$\frac{\left\{\dfrac{\left\{\sum_{I=0}^{\infty}\left[\dfrac{d^{2}}{d\tau^{2}}X_{I}(\tau)\right]\cdot\varepsilon^{I}\right\}}{\left\{\sum_{J=1}^{\infty}\left[\dfrac{d}{d\tau}t_{J}(\tau)\right]\cdot\varepsilon^{J}+1\right\}}+\dfrac{(-1)\cdot\left\{\sum_{J=1}^{\infty}\left[\dfrac{d^{2}}{d\tau^{2}}t_{J}(\tau)\right]\cdot\varepsilon^{J}\right\}\cdot\sum_{I=0}^{\infty}\left[\dfrac{d}{d\tau}X_{I}(\tau)\right]\cdot\varepsilon^{I}}{\left\{\sum_{J=1}^{\infty}\left[\dfrac{d}{d\tau}t_{J}(\tau)\right]\cdot\varepsilon^{J}+1\right\}^{2}}\right\}}{\left\{\sum_{J=1}^{\infty}\left[\dfrac{d}{d\tau}t_{J}(\tau)\right]\cdot\varepsilon^{J}+1\right\}}+\omega^{2}\cdot\sum_{I=0}^{\infty}\varepsilon^{I}\cdot X_{I}(\tau)=\varepsilon\cdot\left[\sum_{I=0}^{\infty}\varepsilon^{I}\cdot X_{I}(\tau)\right]$$

$$\left\{(-2)\cdot\left[\frac{d^2}{d\tau^2}X_0(\tau)\right]\cdot\left[\frac{d}{d\tau}t_1(\tau)\right]+\frac{d^2}{d\tau^2}X_1(\tau)+(-1)\cdot\left[\frac{d^2}{d\tau^2}t_1(\tau)\right]\cdot\left[\frac{d}{d\tau}X_0(\tau)\right]+X_1(\tau)\cdot\omega^2\right\}\cdot\varepsilon+\frac{d^2}{d\tau^2}X_0(\tau)+X_0(\tau)\cdot\omega^2=X_0(\tau)^3 \tag{E2}$$

$$(-2)\cdot\left[\frac{d^2}{d\tau^2}X_0(\tau)\right]\cdot\left[\frac{d}{d\tau}t_1(\tau)\right]+\frac{d^2}{d\tau^2}X_1(\tau)+(-1)\cdot\left[\frac{d^2}{d\tau^2}t_1(\tau)\right]\cdot\left[\frac{d}{d\tau}X_0(\tau)\right]+X_1(\tau)\cdot\omega^2=X_0(\tau)^3 \tag{E3}$$

```
#'E9←SIMPLIFY(DRVDO(SUBSTITUTE(SOLVE(E8,'(X[1](TAU))),'(A*COS(OMEGA*TAU)),
                                                    '(X[0](TAU))),'TAU)) #
```

```
#EDISPLAY('E9)#
```

It is now necessary to substitute an identity for $\cos^3\omega\tau$ and to collect terms on $\sin\omega\tau$ and $\cos\omega\tau$.

```
#'E10←COLLECT(EXPAND(SUBSTITUTE(E9,'((COS(3*OMEGA*TAU)+3*COS(OMEGA*TAU))/4),
                 '((COS(OMEGA*TAU))↑3))),'((SIN(OMEGA*TAU),COS(OMEGA*TAU))))#
#EDISPLAY('E10)#
```

Theoretical considerations require that the coefficients of $\cos\omega\tau$ and $\sin\omega\tau$ must be zero if there is to be a periodic solution for $X_1(\tau)$. From the coefficient of $\sin\omega\tau$ it is apparent that $t_1'(\tau)$ must be some constant C. This constant is determined from the coefficient of $\cos\omega\tau$.

```
#'E11←SIMPLIFY(SOLVE(SUBSTITUTE(!E10,44,'C,'(DRV(TAU,1,:T[1](TAU))))=0,'C))#
```

```
#EDISPLAY('E11)#
```

So $X_0 = A\cos\omega\tau$ and to first order $t = \tau + \varepsilon\frac{3A^2}{8\omega^2}\tau$.

Thus to first order $\tau \approx t(1 - \frac{\varepsilon 3A^2}{8\omega^2})$ and $X_0 \approx A\cos(1 - \frac{\varepsilon 3A^2}{8\omega^2})t$

(E9)

$$\omega^2 \cdot X_1(\tau) + \frac{d^2}{d\tau^2} X_1(\tau)$$

$$= (-2) \cdot \left[\frac{d}{d\tau} t_1(\tau) \right] \cdot \cos(\omega \cdot \tau) \cdot \omega^2 \cdot A + (-1) \cdot \sin(\omega \cdot \tau) \cdot \omega \cdot A \cdot \left[\frac{d^2}{d\tau^2} t_1(\tau) \right] + A^3 \cdot \cos(\omega \cdot \tau)^3$$

(E10)

$$\frac{d^2}{d\tau^2} X_1(\tau) + X_1(\tau) \cdot \omega^2$$

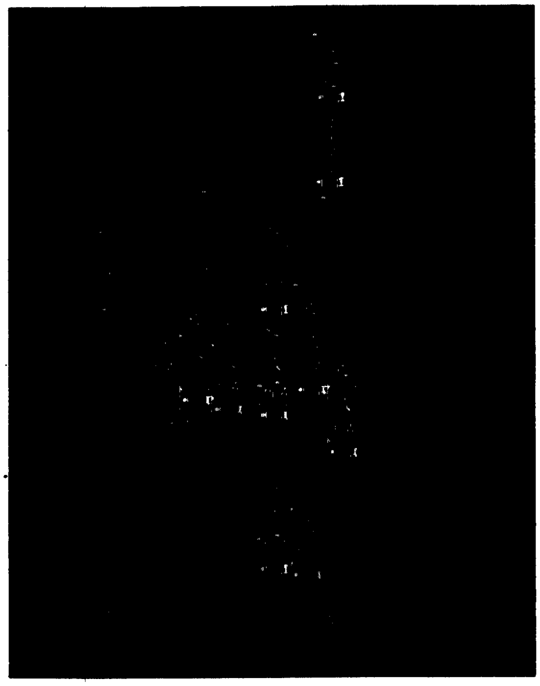
$$= (-1) \cdot \omega \cdot A \cdot \left[\frac{d^2}{d\tau^2} t_1(\tau) \right] \cdot \sin(\omega \cdot \tau) + \left\{ (-2) \cdot \left[\frac{d}{d\tau} t_1(\tau) \right] \cdot \omega^2 \cdot A + \frac{3}{4} \cdot A^3 \right\} \cdot \cos(\omega \cdot \tau) + \frac{1}{4} \cdot \cos(3 \cdot \omega$$

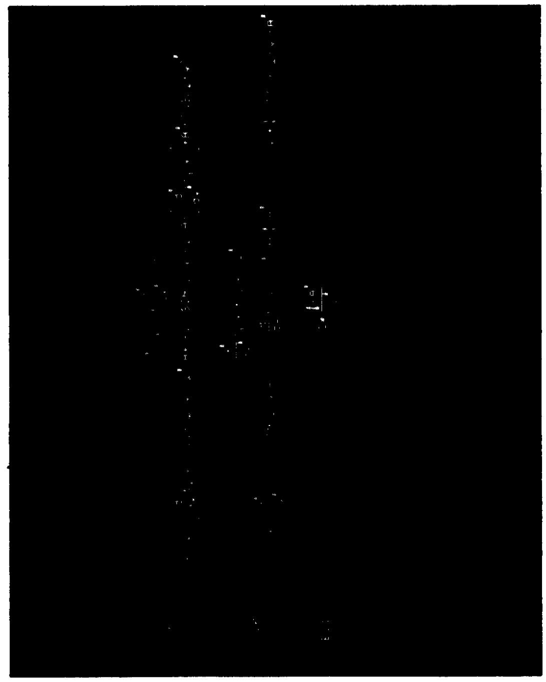
(E11)

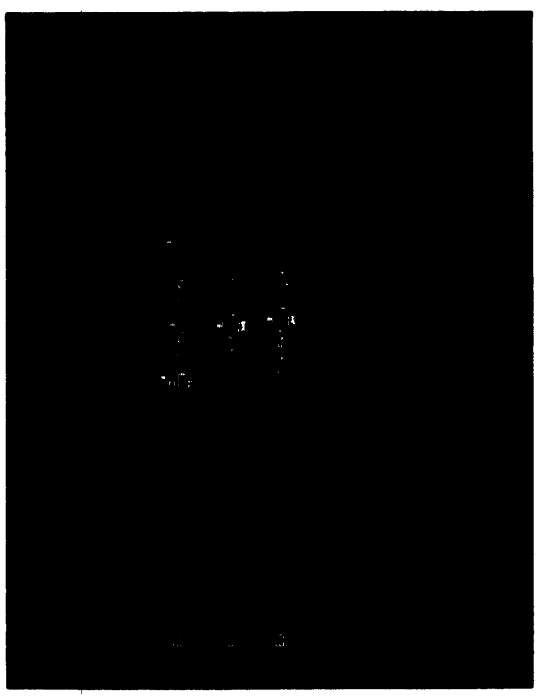
$$C = \frac{\frac{3}{8} \cdot A^2}{\omega^2}$$

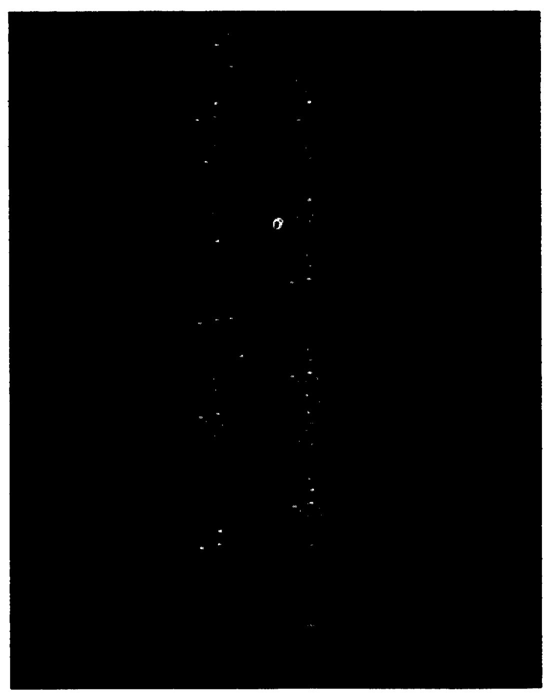
One effect of the nonlinear term is thus seen to be an alteration in the frequency of the zero order term.

In conclusion, note the large number of small steps necessary to solve this problem. These are the result of doing almost the entire solution in the machine. In the case of a small problem such as this, some of these steps could be done by hand, the object here is to illustrate the steps which would be required for a larger problem. This problem also illustrates how rather lengthy intermediate calculations can lead to some rather concise results. The perturbation of the frequency in the zero order function is found to have a simple expression.









III. Plasma Accelerator Electrode Boundary Layers

The second problem is a duplication of the work in the first three sections of chapter three of a 1963 Masters Thesis by J.S. Draper for the M.I.T. Department of Aeronautics and Astronautics. This thesis investigates the laminar compressible boundary layer on the electrode walls of a direct-current crossed field plasma accelerator under very special physical conditions. Many of the assumptions used under these conditions are set forth in the paper "Electrode Boundary Layers in Direct-Current Plasma Accelerators" by Jack L. Kerrebrock in the August 1961 issue of the Journal of Aerospace Sciences. Kerrebrock's paper investigates a solution involving less mathematical manipulation than that undertaken by Masters Thesis student Draper.

In summary, the entire solution procedure is as follows:

1. Write down 5 non-linear partial differential equations:

   Momentum

   State

   Continuity

   Energy

   Electron mobility as a function of temperature.

These equations relate

- U stream velocity
- V lateral velocity
- t temperature
- ρ density
- μ electron mobility
- P pressure

in terms of the independent variables x and y. The constants are:

j current

B magnetic field

C_p specific heat

K compressibility

G conductivity

R gas constant.

2. The absence of variation in the y direction in the free stream is used to find the momentum and state equations there. These two reduced equations are solved for $\frac{dP}{dx}$ which is eliminated from the 5 main equations, since P is not a function of y.

3. The relation $H = C_p t + U^2/2$ is used to substitute derivatives of H for those of t in the energy equation which then becomes an enthalpy equation. t is thus eliminated from this equation. Simplification of the resulting expression requires introduction of the momentum relation. This step is performed because the enthalpy equation has a term proportional to $[1 - \frac{1}{P_r}]$ where P_r is $\frac{C_p \mu}{K}$ and can be approximated as 1, thus eliminating this term.

4. Next, a change of independent and dependent variables is made. The change of independent variables is such that it approximates a similarity transformation for low Mach number. These transformations change x and y to $\bar{x}$ and η. In addition $\frac{U}{U_\infty}$ is defined as f' , $\frac{t}{t_\infty}$ as Θ, and $\frac{H}{H_\infty}$ as g. The momentum and enthalpy equations are transformed, using the continuity and state equations as side conditions. $\bar{x}$ is then changed to M_∞ There result two non-linear differential equations in f and g and their derivatives with respect to η and M_∞

5. f and g are then approximated as $f'(\eta,M_\infty) = b(M_\infty)\eta + c(M_\infty)\eta^2$, $g(\eta,M_\infty) = e(M_\infty)\eta + f(M_\infty)\eta^2$. These approximations are substituted into the two non-linear differential equations. The equations are then integrated with respect to η between the wall and the edge of the velocity boundary layer δu and the edge of the enthropy boundary layer δe respectively. $b(M_\infty)$ and $e(M_\infty)$ are eliminated from the result by the relations $f'(\delta u,M_\infty) = 1$ and $g(\delta e,M_\infty) = 1$. There result two ordinary linear differential equations for the derivatives of δu, c, δe, and f with respect to M_∞.
6. Two more linear differential equations for δu, c, δe, and f are generated by choosing the coefficients of the approximations in step 5 so as to satisfy the momentum and enthalpy equations produced in step 4 exactly at the extremal points $f'' = 0$ and $g' = 0$.
7. The four resulting linear differential equations are solved for the derivatives $\frac{\partial\delta u}{\partial M_\infty}$, $\frac{\partial c}{\partial M_\infty}$, $\frac{\partial\delta e}{\partial M_\infty}$, and $\frac{\partial f}{\partial M_\infty}$ by Gaussian reduction. These four expressions are then numerically integrated with a Runge-Kutta method.

This problem has several interesting features. It is a demonstration of the notation use by workers in this area. The algebraic expressions are of a size difficult to manipulate by hand, but within the capabilities of current machines. The final symbolic result is large; it is difficult to write the corresponding numerical integration program correctly when this result must be input by hand, but here it is developed in the machine and could then be transformed into the required numerical program. Note that the symbolic steps are needed in order to cast the problem in terms of the independent and dependent variables of interest. The problem is characterized by the application of simplifying side conditions and physical assumptions. As such, it involves a number of manipulations for the purpose of expression condensation. This will be apparent from the following step by step reproduction of the first three sections of Chapter III. These steps bring the described solution through the application of the similarity transformation to the momentum equation.

Input the momentum equation:

```
#'D1←'(RHO*(U*DRV(X,1,U) + V*DRV(Y,1,U)) = DRV(Y,1,MU*
          DRV(Y,1,U)) - DRV(X,1,P) + :J*B)#
```

Input the energy equation:

```
#'D2←'(RHO*C[P]*(U*DRV(X,1,:T) + V*DRV(Y,1,:T)) =
         DRV(Y,1,K*DRV(Y,1,:T)) + MU*(DRV(Y,1,U))↑2
         + U*DRV(X,1,P) + :J↑2/SIGMA#
```

The boundary layer solutions must match the free stream solution. The free stream values are indicated by the subscript ∞. At the free stream, there is no variation in the boundary layer with respect to y. To save rewriting, define sets containing the variables to be subscripted.

```
#'D3←'(RHO,SIGMA,U,:T,H)#
#'D4←'(RHO[INF],SIGMA[INF],U[INF],:T[INF],H[INF])#
```

Then in the free stream D1 and D2 become:

```
#'D5←SIMPLIFY(SUBSTITUTE(DRVZERO(D1,'Y),D4,D3))#
#'D6←SIMPLIFY(SUBSTITUTE(DRVZERO(D2,'Y),D4,D3))#
#EDISPLAY('D5)#
#EDISPLAY('D6)#
```

(D5) $$RHO_{\infty} \cdot \left(\frac{d}{dX} U_{\infty} \right) \cdot U_{\infty} = -\frac{dP}{dX} + j \cdot B$$

(D6) $$RHO_{\infty} \cdot C_p \cdot \left(\frac{d}{dX} t_{\infty} \right) \cdot U_{\infty} = U_{\infty} \cdot \left(\frac{dP}{dX} \right) + \frac{j^2}{\sigma_{\infty}}$$

(D1) $$RHO \cdot \left[U \cdot \left(\frac{dU}{dX} \right) + V \cdot \left(\frac{dU}{dY} \right) \right] = \frac{d}{dY} MU \cdot \left(\frac{dU}{dY} \right) - \frac{dP}{dX} + j \cdot B$$

D5 and D6 will now be used to express some of the expressions in D1 and D2 in terms of the free stream quantities. In order to substitute for some of the terms in a sum, the terms must be grouped using the light pen. This is somewhat inconvenient.

```
#EDISPLAY(D1)#
```

Now the last two terms in D1 are replaced by the left side of D5.

```
#'D7←REPLACE ('D1,LEFT(D5), GROUP((!D1,33,!D1,38)))#
#EDISPLAY('D7)#
#'D8←LEFT(D2) = EXPAND(SUBSTITUTE(
        RIGHT(D2), RIGHT(SOLVE(D6,'(DRV(X,1,P)))), '(DRV(X,1,P))))#
#EDISPLAY('D8)#
```

The last two terms in D8 are now put in a factored form.

```
#'D8←REPLACE('D8,FACTOROUT(GROUP((!D8,62,!D8,78)),'(:J↑2/SIGMA[INF]))
                                 ' HOLE)#
#EDISPLAY('D8)#
```

In order to effect a cancellation, equation D8 will now be transformed by replacing the temperature, t, with the enthalpy, H, using the definition

$H = C_p t + U^2/2.$

First $H = C_p t + U^2/2$ is solved for temperature t.

```
#'D9←SOLVE('(H(X,Y) = C[P]*:T(X,Y) + U(X,Y)↑2/2),'(:T(X,Y)))#
```

Now the substitution is made for t and t_∞.

```
#'D10←EXPAND(SUBSTITUTE(D8,(DRV('X,1,RIGHT(D9)),SUBSTITUTE(DRV('X,1,RIGHT(D9)),
          D4,D3),DRV('Y,1,RIGHT(D9))),'( DRV(X,1,:T), DRV(X,1,:T[INF]),
          DRV(Y,1,:T))))#
```

[07] $$RHO \cdot \left[U \cdot \left(\frac{dU}{dX} \right) + V \cdot \left(\frac{dU}{dY} \right) \right] = \frac{d}{dY} MU \cdot \left(\frac{dU}{dY} \right) + RHO_{\infty} \cdot \left(\frac{d}{dX} U_{\infty} \right) \cdot U_{\infty}$$

[08] $$RHO \cdot C_{p} \cdot \left[U \cdot \left(\frac{dt}{dX} \right) + V \cdot \left(\frac{dt}{dY} \right) \right] = \frac{d}{dY} \left(\frac{dt}{dY} \right) \cdot K + \left(\frac{dU}{dY} \right)^{2} \cdot MU + RHO_{\infty} \cdot C_{p} \cdot \left(\frac{d}{dX} t_{\infty} \right) \cdot U - \frac{j^{2} \cdot U}{U_{\infty} \cdot \sigma_{\infty}} + \frac{j^{2}}{\sigma}$$

[08] $$RHO \cdot C_{p} \cdot \left[U \cdot \left(\frac{dt}{dX} \right) + V \cdot \left(\frac{dt}{dY} \right) \right] = \frac{d}{dY} \left(\frac{dt}{dY} \right) \cdot K + \left(\frac{dU}{dY} \right)^{2} \cdot MU + RHO_{\infty} \cdot C_{p} \cdot \left(\frac{d}{dX} t_{\infty} \right) \cdot U + \frac{j^{2} \cdot \left(-\frac{U}{U_{\infty}} + \frac{\sigma_{\infty}}{\sigma} \right)}{\sigma_{\infty}}$$

Now, there are a number of tedious grouping steps.

```
#EDISPLAY('D10)#
```

First, factor RHO out of two of the terms on the left side of the equation.

```
#'D11←REPLACE('D10,FACTOROUT(GROUP((!D10,48,!D10,21)),'RHO),'HOLE)#
#EDISPLAY('D11)#
```

Now factor RHO·U from the other two terms and bring them to the right side.

```
#'D11←BRINGOVER('D11,FACTOROUT(GROUP((!D11,5,!D11,45)),'(RHO*U(X,Y))))#
```

A machine matching operation would be better for the next factoring step which must be done twice when light-pen pointing is used.

```
#EDISPLAY('D11)#
```

The quantity $K/C_p\mu$ is factored out of two of the terms and set equal to $1/P_r$.

```
#'D11←REPLACE('D11,FACTOROUT(!D11,34,'(K/(C[P]*MU)),'(1/P[R])),'HOLE)#
#EDISPLAY('D11)#
#'D11←REPLACE('D11,FACTOROUT(!D11,25,'(K/(C[P]*MU)),'(1/P[R])),'HOLE)#
```

$C_p\mu/K$ is the Prandtl number P_r. It is close to 1, and setting it to 1 will effect a simplification if the identity

$$\mu\left(\frac{dU}{dY}\right)^2 = \frac{d}{dY}\left[\mu \cdot U(X,Y)\frac{d\,U(X,Y)}{dY}\right] - U \cdot \frac{d}{dY}\left[\mu\frac{dU}{dY}\right]$$

is also substituted. This simplification means that the heat conduction away from a point is just equal to the viscous dissipation at that point.

```
#'D11←SIMPLIFY(SUBSTITUTE(D11,'(1,DRV(Y,1,MU*U(X,Y)*DRV(Y,1,
U(X,Y))) -U(X,Y)*DRV(Y,1,MU*DRV(Y,1,U))),'(P[R],MU*(DRV(Y,1,U))↑2)))#
#EDISPLAY('D11)#
```

Next, the substitution in D11 of the right side of equation D7 for its left side effects a nice cancellation.

[n1a] $$-U(X,Y)\cdot\left[\frac{d}{dY}U(X,Y)\right]\cdot U\cdot \dot{RHO}+\left[\frac{d}{dY}H(X,Y)\right]\cdot U\cdot RHO-U(X,Y)\cdot\left[\frac{d}{dX}U(X,Y)\right]\cdot U\cdot RHO+\left[\frac{d}{dX}H(X,Y)\right]\cdot U\cdot RHO$$

$$\frac{d}{dY}\frac{K\cdot U(X,Y)\cdot\left[\frac{d}{dY}U(X,Y)\right]}{C_P}+\frac{d}{dY}\frac{K\cdot\left[\frac{d}{dY}H(X,Y)\right]}{C_P}+MU\cdot\left(\frac{dU}{dY}\right)^2+U\cdot\left[\frac{d}{dX}H_\infty(X,Y)\right]\cdot RHO_\infty$$

$$-U\cdot U_\infty(X,Y)\cdot\left[\frac{d}{dX}U_\infty(X,Y)\right]\cdot RHO_\infty+\frac{j^2}{\sigma}-\frac{U\cdot j^2}{U_\infty\cdot\sigma_\infty}$$

$$(D11)\quad -U(X,Y)\cdot\left[\frac{d}{dY}U(X,Y)\right]\cdot V\cdot RHO+RHO\cdot\left\{\left[\frac{d}{dX}H(X,Y)\right]\cdot U+\left[\frac{d}{dY}H(X,Y)\right]\cdot V\right\}-U(X,Y)\cdot\left[\frac{d}{dX}U(X,Y)\right]\cdot U\cdot RH$$

$$-\frac{d}{dY}\,\frac{K\cdot U(X,Y)\cdot\left[\frac{d}{dY}U(X,Y)\right]}{C_p}+\frac{d}{dY}\,\frac{K\cdot\left[\frac{d}{dY}H(X,Y)\right]}{C_p}+MU\cdot\left(\frac{dU}{dY}\right)^2+U\cdot\left[\frac{d}{dX}H_\infty(X,Y)\right]\cdot RHO_\infty$$

$$-U\cdot U_\infty(X,Y)\cdot\left[\frac{d}{dX}U_\infty(X,Y)\right]\cdot RHO_\infty+\frac{j^2}{\sigma}-\frac{U\cdot j^2}{U_\infty\cdot\sigma_\infty}$$

$$RHO \cdot \left\{ \left[\frac{d}{dX} H(X,Y) \right] \cdot U + \left[\frac{d}{dY} H(X,Y) \right] \cdot V \right\}$$

$$= -\frac{d}{dY} \frac{K \cdot U(X,Y) \cdot \left[\frac{d}{dY} U(X,Y) \right]}{C_p} + \frac{d}{dY} \frac{K \cdot \left[\frac{d}{dY} H(X,Y) \right]}{C_p} + MU \cdot \left(\frac{dU}{dY} \right)^2 + U \cdot \left[\frac{d}{dX} H_\infty(X,Y) \right] \cdot RHO_\infty$$

$$- U \cdot U_\infty(X,Y) \cdot \left[\frac{d}{dX} U_\infty(X,Y) \right] \cdot RHO_\infty + \frac{j^2}{\sigma} - \frac{U \cdot j^2}{U_\infty \cdot \sigma_\infty} - RHO \cdot U(X,Y) \cdot \left\{ -\left[\frac{d}{dY} U(X,Y) \right] \cdot V - \left[\frac{d}{dX} U(X,Y) \right] \cdot U \right\}$$

(011)

$$RHO \cdot \left\{ \left[\frac{d}{dX} H(X,Y) \right] \cdot U + \left[\frac{d}{dY} H(X,Y) \right] \cdot V \right\}$$

$$\frac{d}{dY} \frac{U(X,Y) \cdot \left[\frac{d}{dY} U(X,Y) \right] \cdot MU}{P_r} + \frac{d}{dY} \frac{K \cdot \left[\frac{d}{dY} H(X,Y) \right]}{C_p} + MU \cdot \left(\frac{dU}{dY} \right)^2 + U \cdot \left[\frac{d}{dX} H_\infty(X,Y) \right] \cdot RHO_\infty$$

$$-U \cdot U_\infty(X,Y) \cdot \left[\frac{d}{dX} U_\infty(X,Y) \right] \cdot RHO_\infty + \frac{j^2}{\sigma} - \frac{U \cdot j^2}{U_\infty \cdot \sigma_\infty} - RHO \cdot U(X,Y) \cdot \left\{ -\left[\frac{d}{dY} U(X,Y) \right] \cdot V - \left[\frac{d}{dX} U(X,Y) \right] \cdot U \right\}$$

```
#'D11←SIMPLIFY(REPLACE('D11,(-RIGHT(D7)),GROUP((!D11,117,!D11,122))))#
#'D11←LEFT(D11) = SIMPLIFY(EXPAND(SUBSTITUTE(RIGHT(D11),'(U,U[INF]),'(U,(X,Y),
                  U[INF](X,Y)))))#
#EDISPLAY('D11)#
```

The momentum equation D7 and the enthalpy equation D11 are now ready for the similarity transformation.

The transformations of independent variables are:

```
#'D12←'(XB(X) = ITG(X,0,X,P*U[INF]/(P[0]*U[0])))#
#'D13←'(ETA(X,Y) = (U[INF]/U[0])*((U[0]/(2*NU[0]*XB(X))↑(1/2))*
            ITG(Y,0,Y,RHO/RHO[0]))#
```

Next, to compute the required differentials.

First $\frac{1}{dX} = \frac{d\eta}{dX} \cdot \frac{1}{d\eta} + \frac{d\overline{X}}{dX} \cdot \frac{1}{d\overline{X}}$

```
#'D14←DRV('X,1,RIGHT(D13))*'(DEL(ETA))+DRV('X,1,RIGHT(D12))*'DEL(XB))#
#EDISPLAY('D14)#
```

D14 now contains the expression for η as a factor.

```
#'D14←REPLACE('D14,FACTOROUT(!D14,4,RIGHT(D13),'(ETA(X,Y))),'HOLE)#
#EDISPLAY('D14)#
```

η is now substituted for its definition and $(P \cdot U_\infty)/(P_0 \cdot U_0)$ is factored out.

```
#'D14←SIMPLIFY(FACTOROUT(SUBSTITUTE(D14,DRV('X,1,RIGHT(D12)),
            '(DRV(X,1,XB(X)))),'P*U[INF]/(P[0]*U[0]))))#
#EDISPLAY('D14)#
```

The differential for 1/dY is

```
#'D15←DRV('Y,1,RIGHT(D13))*'(DEL(ETA))#
```

$$(D11)\quad RHO\cdot\left\{\left[\frac{d}{dX}H(X,Y)\right]\cdot U+\left[\frac{d}{dY}H(X,Y)\right]\cdot V\right\}$$

$$=\frac{d}{dY}\left[\frac{d}{dY}H(X,Y)\right]\cdot MU-U(X,Y)\cdot\left[\frac{d}{dY}MU\cdot\left(\frac{dU}{dY}\right)\right]+U\cdot\left[\frac{d}{dX}H_{\infty}(X,Y)\right]\cdot RHO_{\infty}-U\cdot U_{\infty}(X,Y)\cdot\left[\frac{d}{dX}U_{\infty}(X,Y)\right]\cdot RHO$$

$$+\frac{J^2}{\sigma}-\frac{U\cdot J^2}{U_{\infty}\cdot\sigma_{\infty}}-RHO\cdot U(X,Y)\cdot\left\{-\left[\frac{d}{dY}U(X,Y)\right]\cdot V-\left[\frac{d}{dX}U(X,Y)\right]\cdot U\right\}$$

$$(D11)\quad RHO\cdot\left\{\left[\frac{d}{dX}H(X,Y)\right]\cdot U+\left[\frac{d}{dY}H(X,Y)\right]\cdot V\right\}=\frac{d}{dY}MU\cdot\left[\frac{d}{dY}H(X,Y)\right]+RHO_{\infty}\cdot\left[\frac{d}{dX}H_{\infty}(X,Y)\right]\cdot U+\frac{J^2}{\sigma}-\frac{J^2\cdot U}{\sigma_{\infty}\cdot U_{\infty}}$$

(П14)
$$-\frac{\left[\frac{d}{dX}XB(X)\right]\cdot NU_0\cdot 2\cdot U_0\cdot\left[\frac{U_0}{2\cdot NU_0\cdot XB(X)}\right]^{\frac{1}{2}-1}\cdot U_\infty\cdot\left(\int_0^Y \frac{RHO}{RHO_0}\,dY\right)\cdot DEL(ETA)}{\left[2\cdot NU_0\cdot XB(X)\right]^2\cdot 2\cdot U_0}+\frac{P\cdot U_\infty\cdot DEL(XB)}{P_0\cdot U_0}$$

(П14)
$$-\frac{\frac{1}{2}\cdot ETA(X,Y)\cdot\left[\frac{d}{dX}XB(X)\right]\cdot DEL(ETA)}{XB(X)}+\frac{P\cdot U_\infty\cdot DEL(XB)}{P_0\cdot U_0}$$

```
#EDISPLAY('D15)#
```

Expressions D14 and D15 are the required differentials for $\frac{d}{dX}$ and $\frac{d}{dY}$ respectively.

Now to transform the momentum equation D7. First, substitute for the differentials of the independent variables and the new normalized dependent variable f, defined by $U/U_\infty = f'(\overline{X}, \eta)$.

```
#'D16←SIMPLIFY(SUBSTITUTE(DELSUBST(DELSUBST(D7,'(DEL(X)),D14),
                     '(DEL(Y)),D15),'(U[INF](XB)*DRV(ETA,1,
                     :F(XB,ETA)),U[INF](XB),XB,ETA),'(U,U[INF],XB(X),ETA(X,Y))))#
```

Now assume $\mu = \mu_0 t/t_0$.

```
#'D17←SIMPLIFY(SUBSTITUTE('D16,'(MU[0]*:T/:T[0],'MU))#
```

Use the equation of state to recognize that

$$\frac{d}{d\eta}(\rho T) = \frac{dP}{d\eta} = 0$$

since P is taken independent of Y. A substitution for P should therefore beemade.

```
#EDISPLAY('D17)#
```

The machine responds with "EXPRESSION TOO LARGE"

Therefore, the left side of equation D17 will be treated first, the substitution for P will be deferred.

```
#'D18←LEFT(D17)#
```

An expression for V will now be developed and substituted into D18. It was shown in Chapter II of the thesis that the continuity equation yields

$$V = -\frac{\rho_0}{\rho}\frac{d}{dX}(2U_0\nu_0\,\overline{X})^{\frac{1}{2}}f$$

$$\frac{P \cdot U_{\omega} \cdot \left[\frac{\left(-\frac{1}{2}\right) \cdot ETA(X,Y) \cdot DEL(ETA)}{XB(X)} + DEL(XB) \right]}{P_{0} \cdot U_{0}}$$ [illegible]

$$\frac{RHO \cdot \left[\frac{U_{0}}{2 \cdot NU_{0} \cdot XB(X)} \right]^{\frac{1}{2}} \cdot U_{\omega} \cdot DEL(ETA)}{RHO_{0} \cdot U_{0}}$$ [illegible]

Enter this expression.

```
#'D19←'(-(RHO[0]/RHO)*DRV(X,1,((2*U[0]*NU[0]*XB↑(1/2))*:F(XB.ETA)))#
```

Now substitute the new independent variables:

```
#'D20←SUBSTITUTE(DELSUBST(D19,'(DEL(X)),D14),'(U[INF](XB),XB,ETA),
                '(U[INF],XB(X),ETA(X,Y)))#
```

Now, substituting this expression for V into the partially transformed left side of the momentum equation, D18, the entire expression is differentiated as far as possible, expanded and simplified.

```
#'D21←EXPAND(DRVDO(DRVDO(SUBSTITUTE(D18,D20,'V),'ETA),'XB))#
```

Now some shorthand will be introduced to make D21 easier to read.

```
#'D22←'(:F,U,F1,F2,F01,F11,U[INF])
#'D23←'(:F(XB,ETA),U(XB),DRV(ETA,1,:F),DRV(ETA,2,:F),DRV(XB,1,:F),DRV
             (ETA,1,XB,1,:F),U[INF](XB))#
#'D21←SUBSTITUTE(D21,D22,D23)#
#EDISPLAY('D21)#
```

It is now convenient to factor a large coefficient from each term and set it equal to 1 since it will also be factored from the other side and set to 1 there.

```
#'D24←SIMPLIFY(FACTOROUT(D21,'(RHO*U[INF]↑3*P/(2*P[0]*U[0]*XB)),1))#
#EDISPLAY('D24)#
```

Returning to the right side of D17.

```
#'D25←RIGHT(D17)#
#EDISPLAY('D25)#
```

The deferred substitution for P is now made.

```
#'D27←REPLACE('D25,FACTOROUT(!D25,44,'((RHO*:T)/(RHO[0]*:T[0])),'(P/P[0])),'HOLE)#
```

(n21) $$-\frac{U_\infty^3 \cdot F2 \cdot RHO \cdot F01 \cdot P}{U_\infty \cdot P_\infty} + \frac{\left(-\frac{1}{2}\right) \cdot U_\infty^3 \cdot F2 \cdot RHO \cdot f \cdot P}{XB \cdot U_\infty \cdot P_\infty} + \frac{F1^2 \cdot \left(\frac{d}{dXB} U_\infty\right) \cdot P \cdot U_\infty^2 \cdot RHO}{U_\infty \cdot P_\infty} + \frac{U_\infty^3 \cdot F11 \cdot P \cdot F1 \cdot RHO}{U_\infty \cdot P_\infty}$$

(n24) $$-2 \cdot F2 \cdot F01 \cdot XB - F2 \cdot f + \frac{2 \cdot F1^2 \cdot \left(\frac{d}{dXB} U_\infty\right) \cdot XB}{U_\infty} + 2 \cdot F11 \cdot F1 \cdot XB$$

$$RHO \cdot 2^{-\frac{1}{2}} \cdot U_0^{-\frac{1}{2}} \cdot NU_0^{-\frac{1}{2}} \cdot XB^{-\frac{1}{2}} \cdot U_\infty(XB)$$

$$\frac{\cdot \left\{ \frac{d}{dETA} \frac{MU_0 \cdot t \cdot RHO \cdot 2^{-\frac{1}{2}} \cdot U_0^{-\frac{1}{2}} \cdot NU_0^{-\frac{1}{2}} \cdot XB^{-\frac{1}{2}} \cdot U_\infty(XB) \cdot \left\{ \frac{d}{dETA} U_\infty(XB) \cdot \left[\frac{d}{dETA} f(XB, ETA) \right] \right\}}{t_0 \cdot RHO_0} \right\}}{RHO_0} \quad (0.25)$$

$$+ \frac{RHO_\infty \cdot P \cdot U_\infty(XB)^2 \cdot \left\{ \frac{\left(-\frac{1}{2}\right) \cdot ETA \cdot \left[\frac{d}{dETA} U_\infty(XB) \right]}{XB} + \frac{d}{dXB} U_\infty(XB) \right\}}{P_0 \cdot U_0}$$

The expression is now differentiated as far as possible, as was the left side.

```
#'D28←EXPAND(DRVDO(DRVDO(D27,'ETA),'XB))#
```

The simplifying notation is substituted.

```
#'D28←SUBSTITUTE(D28,D22,D23)#
#EDISPLAY('D28)#
```

Finally the large factor is removed from each term as it=was from the right side.

```
#'D29←SIMPLIFY(FACTOROUT(D28,'(RHO*U[INF]↑3*P/(2*P[0]*U[0]*XB)),1))#
```

The two sides of the transformed momentum equation are now recombined.

```
#'D30←SIMPLIFY(D24-D29) = 0#
#EDISPLAY('D30)#
```

Using some more relations developed earlier in the thesis, a final substitution will be made, from the equation of state at constant pressure.

$$\frac{\rho_\infty}{\rho} = \frac{t}{t_\infty} \quad \text{but} \quad \frac{t}{t_\infty} = \Theta \quad \text{so} \quad \frac{\rho_\infty}{\rho} = \Theta$$, and from Chapter II,

$$\Theta = (\Theta_s - \Theta_w)g + \Theta_w - (\Theta_s - 1)f^2$$

$$\Theta_s = 1 + \frac{\gamma-1}{2} \cdot M_\infty^2$$

The final transformed equation is:

```
#'D31←SIMPLIFY(SUBSTITUTE(REPLACE('D30,FACTOROUT(!D30,38,'(RHO[INF]/RHO),
          '((THETA[S] - THETA[W])*G + THETA[W] - (THETA[S] - 1)*F1↑2)),'HOLE),
          '(1 + (GAMMA - 1)*M[INF]↑2/2),'(THETA[S])))#
#EDISPLAY('D31)#
```

(D28) $$\frac{\frac{1}{2}\cdot P\cdot MU_{\theta}\cdot U_{\infty}^{3}\cdot\left(\frac{d^{3}f}{dETA^{3}}\right)\cdot RHO}{RHO_{\theta}\cdot P_{\theta}\cdot U_{\theta}\cdot NU_{\theta}\cdot XB}+\frac{\left(\frac{d}{dXB}U_{\infty}\right)\cdot U_{\infty}^{2}\cdot P\cdot RHO_{\infty}}{U_{\theta}\cdot P_{\theta}}$$

(D30) $$-2\cdot F2\cdot F01\cdot XB-F2\cdot f+\frac{2\cdot F1^{2}\cdot\left(\frac{d}{dXB}U_{\infty}\right)\cdot XB}{U_{\infty}}+2\cdot F11\cdot F1\cdot XB-\frac{2\cdot\left(\frac{d}{dXB}U_{\infty}\right)\cdot RHO_{\infty}\cdot XB}{U_{\infty}\cdot RHO}-\frac{MU_{\theta}\cdot\left(\frac{d^{3}f}{dETA^{3}}\right)}{RHO_{\theta}\cdot NU_{\theta}}=0$$

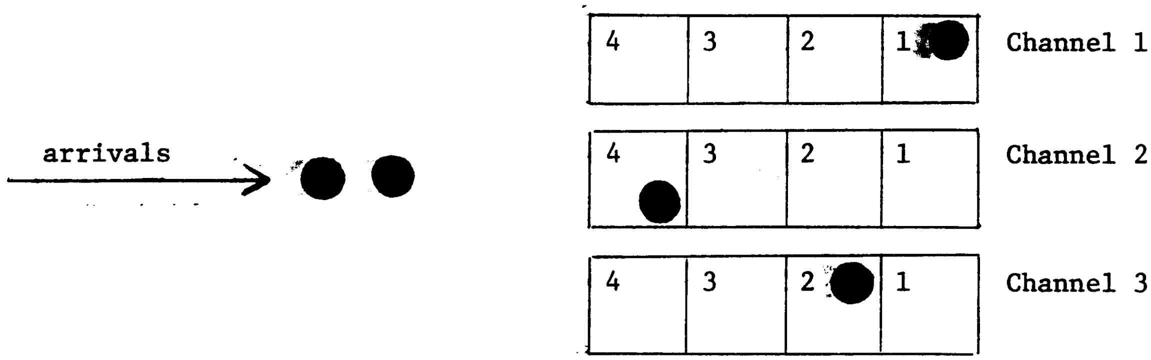
(D31) $$-2\cdot F2\cdot F01\cdot XB-F2\cdot f+\frac{2\cdot F1^{2}\cdot\left(\frac{d}{dXB}U_{\infty}\right)\cdot XB}{U_{\infty}}+2\cdot F11\cdot F1\cdot XB$$

$$-\frac{2\cdot\left\{\left[\frac{1}{2}\cdot(GAMMA-1)\cdot M_{\infty}^{2}-\theta_{u}+1\right]\cdot G+\theta_{u}+\left(-\frac{1}{2}\right)\cdot M_{\infty}^{2}\cdot(GAMMA-1)\cdot F1^{2}\right\}\cdot\left(\frac{d}{dXB}U_{\infty}\right)\cdot XB}{U_{\infty}}-\frac{MU_{\theta}\cdot\left(\frac{d^{3}f}{dETA^{3}}\right)}{RHO_{\theta}\cdot NU_{\theta}}=0$$

In conclusion, since physical approximations are involved in the expression condensation, close man-machine interaction is required. The greatest draw backs to sufficient interaction are the input notation and the lack of sufficient facilities for abbreviation on output.

IV. A Multiple Channel Queueing Problem

The third problem is taken from Interim Technical Report No. 13 of the M.I.T. Center for Operations Research titled "A Class of Queueing Problems". This work is a 1955 Doctor's Thesis by H.N. Garber. The queueing situation shown by the example in Figure 1 is treated.



Arrival times at the queueing complex are exponentially distributed with mean λ. Arrivals enter any channel which becomes vacant and progress through several phases of service with exponentially distributed service times with mean $k\mu$. There are M channels and k phases of service in the general case. It is desired to find the probability distribution of the number of units in the system. The solution can be carried a number of steps for the general case. A set of equations relating the probabilities of the states are written, where a state of the system is taken as a certain number N in the system and a description of which phases of which channels are occupied. Each equation is then multiplied by the appropriate variables so that the summation of the equations will yield an equation with the generating function for the state probabilities as its left side. Unfortunately, the generating function also appears in a summation on the right side. The independent variables of the generating function are constrained to make each term of this summation equal to zero. This constraint is expressed in a change of independent variables. It is then shown that a summation over values of the new independent variables will yield the old generating function.

This generating function still involves the state probabilities for a partially full system as constants to be determined. There are a number of relations which can be used to determine these initial probabilities. As the third problem this evaluation will be carried out for the case of two channels with two phases of service.

The source of expression complexity is different in this problem. In the first problem complexity arose because a small parameter allowed an unsolvable problem to be split into a spectrum of solvable ones. In the second problem complexity was the result of the number of important phenomena in the physical process being described. Here, complexity is the result of the number of states in the process being described, all of which are very similar. In this problem there would be the best chance for reduction in complexity through proper notation.

This problem has been included so that the reader can better refine his intuitive notion of the types of mathematical operations and notation needed to solve problems in different areas of applied mathematics. Summation expansion, function evaluation, limits, and some more grouping operations are introduced.

The function in the transformed variables is:

```
#'Q1←'(K(Z,Q[1],Q[2]) = ((Z↑(2*:K + 1))*SUM(R,1,:K,(Z↑R)*
        A[R] * (:E↑(2*PI*I*Q[1]*R/:K)+:E↑(2*PI*I*Q[2]*R/:K))))/
        (2*(Z*W(Z↑:K) - ALPHA(Q[1],Q[2]))))#
```

Where W, α, and A_r are defined by:

```
#'Q2←'(W(Z↑:K) = 1 + THETA - THETA*Z↑:K)#
#'Q3←'(ALPHA(Q[1],Q[2]) = (:E↑(-2*PI*I*Q[1]/:K)+
        :E↑(-2*PI*I*Q[2]/:K))/2)#
#'Q4←'(A[R](Z) = P(1,R+1,0) - (2*W(Z↑:K) - 1)*P(1,R,0))#
#EDISPLAY('Q1)#
```

It is useful to have the denominator of Q1 contain only powers of Z^k. This is done by using the identity $(X-Y)(X^{k-1} \ldots Y^{k-1}) = (X^k - Y^k)$. At present the system contains no operators for achieving this goal, so it must be done by brute force.

```
#'Q5←LEFT(Q1) = '(SUM(J,0,:K-1,(Z*W(Z↑:K))↑J*ALPHA(Q[1],Q[2]↑(:K-1-J)))*
             SUBSTITUTE(RIGHT(Q1), '((Z↑:K)*(W(Z↑:K)↑:K) - ALPHA(Q[1],Q[2])↑:K)
             '(Z*W(Z↑:K) - ALPHA(Q[1],Q[2])))#
#EDISPLAY('Q5)*
```

Now substitute 2 for the number of phases of service, k. Then expand the summations.

```
#'Q6←SIMPLIFY(ALLSUMEXPAND(SIMPLIFY(SUBSTITUTE(Q5,2,':K))))#
#EDISPLAY('Q6)#
```

(01) $$K(Z, O_1, O_2) = \frac{Z^{2\cdot k+1} \cdot \sum_{R=1}^{k} Z^{R} \cdot A_R \cdot \left(e^{\frac{2\cdot\pi\cdot I\cdot O_1\cdot R}{k}} + e^{\frac{2\cdot\pi\cdot I\cdot O_2\cdot R}{k}} \right)}{2 \cdot \left[Z \cdot W\left(Z^{k}\right) - ALPHA\left(O_1, O_2\right) \right]}$$

(02) $$W\left(Z^{k}\right) = \theta - \theta \cdot Z^{k} + 1$$

(05)

$$K(Z,Q_1,Q_2)=\frac{\frac{1}{2}\cdot Z^{2\cdot k+1}\cdot\left\{\sum_{J=0}^{k-1}\left[Z\cdot W\left(Z^k\right)\right]^J\cdot ALPHA(Q_1,Q_2)^{k-1-J}\right\}\cdot\sum_{R=1}^{k}Z^R\cdot A_R\cdot\left(e^{\frac{2\cdot\pi\cdot I\cdot Q_1\cdot R}{k}}+e^{\frac{2\cdot\pi\cdot I\cdot Q_2\cdot R}{k}}\right)}{\left[Z^k\cdot W\left(Z^k\right)^k-ALPHA(Q_1,Q_2)^k\right]}$$

(06)

$$K(Z,Q_1,Q_2)=\frac{\frac{1}{2}\cdot\left[ALPHA(Q_1,Q_2)+Z\cdot W\left(Z^2\right)\right]\cdot Z^5\cdot\left[Z\cdot A_1\cdot\left(e^{\pi\cdot I\cdot Q_1}+e^{\pi\cdot I\cdot Q_2}\right)+Z^2\cdot A_2\cdot\left(e^{2\cdot\pi\cdot I\cdot Q_1}+e^{2\cdot\pi\cdot I\cdot Q_2}\right)\right]}{\left[Z^2\cdot W\left(Z^2\right)^2-ALPHA(Q_1,Q_2)^2\right]}$$

The substitution k = 2 is made in all the initial equations as well since they will be used several times.

```
#'Q1←SIMPLIFY(SUBSTITUTE(Q1,2,':K))#
#'Q2←SIMPLIFY(SUBSTITUTE(Q2,2,':K))#
#'Q3←SIMPLIFY(SUBSTITUTE(Q3,2,':K))#
#'Q4←SIMPLIFY(SUBSTITUTE(Q4,2,':K))#
```

Now write down an expression for the generating function as a summation of Q6 over the transformed independent variables, the q's.

```
#Q7←'(H(Z↑2,V[1],V[2]) = (1/4)*SUM(Q[1],0,1,
      SUM(Q[2],0,1, ((V[1]/Z)*(-1)↑Q[1] + V[1]↑2/(Z↑2))*
      ((V[2]/Z)*(-1)↑Q[2] + V[2]↑2/(Z↑2))*K(Z,Q[1],Q[2]))))#
#EDISPLAY('Q7)#
```

Now expanding the summations, substituting the appropriate values of Q6, Q3, and k, and simplifying.

```
#Q8←SIMPLIFY(EVALUATE(ALLSUMEXPAND(Q7),(Q6,Q3)))#
#EDISPLAY('Q8)#
```

To evaluate p(1,1,0) and P(1,2,0) certain known conditions are next imposed. Q8 is known to have a zero of order four in Z for all values of V1 and V2, so one would like to collect terms on Z^4. The simplest way to explore this would be to expand Q8 and collect terms on Z^4. Unfortunately, this leads to roughly a sixteen-fold growth in expression size and to memory overflow.

Inspection of Q8 shows that it might be rearranged while in factored form. As the first step, subexpressions which are polynomials in Z, V_1, or

(07) $$H\left(Z^2, U_1, U_2\right) = \frac{1 \cdot \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \left[\frac{U_1 \cdot (-1)^{a_1}}{Z} + \frac{U_1^2}{Z^2}\right] \cdot \left[\frac{U_2 \cdot (-1)^{a_2}}{Z} + \frac{U_2^2}{Z^2}\right] \cdot K\left(Z, a_1, a_2\right)}{4}$$

(08) $$H\left(Z^2, U_1, U_2\right)$$

$$-\frac{1}{4} \cdot \left\{ \frac{\frac{1}{2} \cdot \left(\frac{U_1}{Z} + \frac{U_1^2}{Z^2}\right) \cdot \left(\frac{U_2}{Z} + \frac{U_2^2}{Z^2}\right) \cdot \left[Z \cdot W\left(Z^2\right) + 1\right] \cdot Z^5 \cdot \left(2 \cdot Z \cdot A_1 + 2 \cdot Z^2 \cdot A_2\right)}{\left[Z^2 \cdot W\left(Z^2\right) - 1\right]^2} + \frac{\left(\frac{U_1}{Z} + \frac{U_1^2}{Z^2}\right) \cdot \left(-\frac{U_2}{Z} + \frac{U_2^2}{Z^2}\right) \cdot Z^6 \cdot A_2}{W\left(Z^2\right)} \right.$$

$$\left. + \frac{\left(-\frac{U_1}{Z} + \frac{U_1^2}{Z^2}\right) \cdot \left(\frac{U_2}{Z} + \frac{U_2^2}{Z^2}\right) \cdot Z^6 \cdot A_2}{W\left(Z^2\right)} + \frac{\frac{1}{2} \cdot \left(-\frac{U_1}{Z} + \frac{U_1^2}{Z^2}\right) \cdot \left(-\frac{U_2}{Z} + \frac{U_2^2}{Z^2}\right) \cdot \left[Z \cdot W\left(Z^2\right) - 1\right] \cdot Z^5 \cdot \left(-2 \cdot Z \cdot A_1 + 2 \cdot Z^2 \cdot A_2\right)}{\left[Z^2 \cdot W\left(Z^2\right) - 1\right]^2} \right.$$

V_2, have factors of Z, V_1, or V_2 removed so that their lowest order term is of zero order in these.

```
#'Q81←SIMPLIFY(NORMPOLY(NORMPOLY(NORMPOLY(Q8,'Z),'(V[1])),'(V[2])))#
#EDISPLAY('Q81)#
```

Next the center two terms are combined.

```
#'Q82←REPLACE('Q81,FACTOROUT(GROUP((!Q81,85,!Q81,108)),
        '(A[2]/W(Z↑2))),'HOLE)#
#EDISPLAY('Q82)#
```

The resulting term is then expanded.

```
#'Q83←REPLACE('Q82, EXPAND(!Q82,90)'HOLE)#
#EDISPLAY('Q83)#
```

Now the other two terms are combined.

```
#'Q84←REPLACE('Q83,FACTOROUT(GROUP((!Q83,111,!Q83,30)),1/Q83!70),'HOLE)#
#EDISPLAY('Q84)#  EXPRESSION TOO LARGE
```

Q84 will not display, so it is reduced in size by naming a subpart.

```
#'Q85←SPLIT(Q84)#
#EDISPLAY('Q85)#
```

Next the numerator of the larger term in Q85 is arranged on powers of Z.

```
#'Q86←REPLACE('Q85,COLLECT(EXPAND(SUBSTITUTE(!Q85,31,F2,'F2)),'Z),'HOLE)#
#EDISPLAY('Q86)#          (EXPRESSION TOO LARGE)
#'Q87←SPLIT(Q86)#
#EDISPLAY('Q87)#
```

Forming a term from the renamed parts of Q87 one has for the coefficient of Z^2 in Q87.

```
#'Q88←SIMPLIFY(F2 + F3 + F4 + '(2*A1) + F5 + F6)#
```

[0081]

$$H\left(Z^2, V_1, V_2\right)$$

$$\frac{1}{4} \cdot Z^2 \cdot V_1 \cdot V_2$$

$$\left\{ \frac{1 \cdot (Z+V_1) \cdot (Z+V_2) \cdot \left[Z \cdot W\left(Z^2\right)+1\right] \cdot (2 \cdot A_1 + 2 \cdot Z \cdot A_2)}{\left[Z^2 \cdot W\left(Z^2\right)-1\right]^2} + \frac{(Z+V_1) \cdot (-Z+V_2) \cdot A_2}{W\left(Z^2\right)} + \frac{(-Z+V_1) \cdot (Z+V_2) \cdot A_2}{W\left(Z^2\right)} \right\}$$

$$+ \frac{\frac{1}{2} \cdot (-Z+V_1) \cdot (-Z+V_2) \cdot \left[Z \cdot W\left(Z^2\right)-1\right] \cdot (-2 \cdot A_1 + 2 \cdot Z \cdot A_2)}{\left[Z^2 \cdot W\left(Z^2\right)-1\right]^2}$$

$$H\left(Z^2, U_1, U_2\right)$$

$$\frac{1}{4} \cdot Z^2 \cdot U_1 \cdot U_2 \cdot \left\{ \frac{\frac{1}{2} \cdot (Z+U_1) \cdot (Z+U_2) \cdot \left[Z \cdot W\left(Z^2\right)+1\right] \cdot (2 \cdot A_1 + 2 \cdot Z \cdot A_2)}{\left[Z^2 \cdot W\left(Z^2\right)-1\right]^2} + \frac{A_2 \cdot \left[(Z+U_1) \cdot (-Z+U_2) + (-Z+U_1) \cdot (Z+U_2)\right]}{W\left(Z^2\right)} \right.$$

$$\left. + \frac{\frac{1}{2} \cdot (-Z+U_1) \cdot (-Z+U_2) \cdot \left[Z \cdot W\left(Z^2\right)-1\right] \cdot (-2 \cdot A_1 + 2 \cdot Z \cdot A_2)}{\left[Z^2 \cdot W\left(Z^2\right)-1\right]^2} \right\}$$

[08[illegible]] $$H\left(Z^2, U_1, U_2\right) = \frac{1}{4}\cdot Z^2\cdot U_1\cdot U_2\cdot\left\{\frac{\frac{1}{2}\cdot\left(Z+U_1\right)\cdot\left(Z+U_2\right)\cdot\left[Z\cdot W\left(Z^2\right)+1\right]\cdot\left(2\cdot A_1+2\cdot Z\cdot A_2\right)}{\left[Z^2\cdot W\left(Z^2\right)-1\right]^2}+\frac{A_2\cdot\left(2\cdot U_2\cdot U_1-2\cdot Z^2\right)}{W\left(Z^2\right)}\right.$$

$$\left.+\frac{\frac{1}{2}\cdot\left(-Z+U_1\right)\cdot\left(-Z+U_2\right)\cdot\left[Z\cdot W\left(Z^2\right)-1\right]\cdot\left(-2\cdot A_1+2\cdot Z\cdot A_2\right)}{\left[Z^2\cdot W\left(Z^2\right)-1\right]^2}\right\}$$

[08[illegible]] $$H\left(Z^2, U_1, U_2\right)$$

$$= \frac{1}{4}\cdot Z^2\cdot U_1\cdot U_2\cdot\left\{\frac{\left\{\frac{1}{2}\cdot\left(Z+U_1\right)\cdot\left(Z+U_2\right)\cdot\left[Z\cdot W\left(Z^2\right)+1\right]\cdot\left(2\cdot A_1+2\cdot Z\cdot A_2\right)+F2\right\}}{\left[Z^2\cdot W\left(Z^2\right)-1\right]^2}+\frac{A_2\cdot\left(2\cdot U_2\cdot U_1-2\cdot Z^2\right)}{W\left(Z^2\right)}\right\}$$

```
#EDISPLAY('Q88)#
```

Looking at equation Q87, one can see that in a double power series expansion in V_1 and V_2, only the coefficient of $V_1^2 V_2^2$ does not have a zero of order 4 at Z = 0. Setting this coefficient equal to zero one obtains by inspection of Q87:

$$\frac{A_2}{2W(Z^2)} + \frac{A_1 + Z^2 \cdot W(Z^2) \cdot A_2}{2 \cdot (Z^2 \cdot W^2(Z^2) - 1)} = 0$$

Enter the equation into the machine.

```
#'Q9←'(A[2]/(2*W(Z↑2)) + (A[1] + Z↑2*W(Z↑2)*A[2])/
        (2*(Z↑2*(W(Z↑2))↑2 -1)) = 0)#
```

Evaluating W, A_1 and A_2 one obtains at Z = 0

```
#'Q10←SIMPLIFY(SUBSTITUTE(EVALUATE(Q9,(Q4,Q2)),0,'Z))#
#EDISPLAY('Q10)#
```

Recognizing that by definition P(1,3,0) = P(1,1,0) this equation can be solved for P(1,2,0).

```
#'Q11←SIMPLIFY(SOLVE(SUBSTITUTE(Q10,'(P(1,1,0)),'(P(1,3,0))),'(P(1,2,0))))#
#EDISPLAY('Q11)#
```

Q11 can be simplified somewhat.

```
#'Q11←LEFT(Q11) = (-SIMPLIFY(FACTOROUT(!Q11,11,!Q11,29)/
        FACTOROUT(!Q11,43,!Q11,49)))#
#EDISPLAY('Q11)#
```

(087)

$$H\left(Z^2, U_1, U_2\right)$$

$$-\frac{1}{4}\cdot Z^2\cdot U_1\cdot U_2\cdot\left\{\frac{\left[\left(F2+F3+F4+2\cdot A_1+F5+F6\right)\cdot Z^2+2\cdot W\left(Z^2\right)\cdot A_2\cdot Z^4+2\cdot A_1\cdot U_2\cdot U_1\right]}{\left[Z^2\cdot W\left(Z^2\right)-1\right]^2}+\frac{A_2\cdot\left(2\cdot U_2\cdot U_1-2\cdot Z^2\right)}{W\left(Z^2\right)}\right\}$$

(088)

$$2\cdot U_2\cdot A_2+2\cdot U_1\cdot A_2+2\cdot W\left(Z^2\right)\cdot U_2\cdot U_1\cdot A_2+2\cdot B_1+2\cdot U_2\cdot W\left(Z^2\right)\cdot A_1+2\cdot U_1\cdot W\left(Z^2\right)\cdot A_1$$

(010)

$$\frac{\frac{1}{2}\cdot\left[P(1,3,0)-(2\cdot\theta+1)\cdot P(1,2,0)\right]}{(\theta+1)}+\frac{1}{2}\cdot(2\cdot\theta+1)\cdot P(1,1,0)+\left(-\frac{1}{2}\right)\cdot P(1,2,0)=0$$

(011)

$$P(1,2,0)=-\frac{\left[\frac{1}{2}\cdot P(1,1,0)+P(1,1,0)\cdot\theta+\frac{\frac{1}{2}\cdot P(1,1,0)}{(\theta+1)}\right]}{\left[-\frac{1}{2}+\frac{\left(-\frac{1}{2}\right)}{(\theta+1)}-\frac{\theta}{(\theta+1)}\right]}$$

Let $u = z^2$, the generating function for the unconditional state probabilities $G(u) = \sum_{n=0}^{\infty} u^n p(n)$ can now be written. There are special case terms for n = 1, and n = 2. The other terms are found from H(u,1,1). The most compact formula for H is Q86.

```
#'Q12←'(G(U)) = '(P(0,0,0)) + 2*'U*('(P(1,1,0)) + RIGHT(Q11))
                + SIMPLIFY(SUBSTITUTE(EVALUATE(RIGHT(Q86), (Q2, Q4)),
                          '(U↑(1/2), 1,1),'(Z,
                                    V[1],V[2])))#
```

The original transition equations five p(1,1,0) = θp(0,0,0), anticipating that p(1,3,0) may also occur, substitute p(1,3,0) = p(1,1,0) = θp(0,0,0). Then use Q11 to eliminate P(1,1,0).

```
#'Q13←SUBSTITUTE(Q12, RIGHT(Q11), '(P(1,2,0)))#

#'Q14←SIMPLIFY(SUBSTITUTE(Q13, '(THETA*P(0,0,0), THETA*P(0,0,0)),
                          '(P(1,3,0), P(1,1,0))))#
```

Q14 contains only the unknown P(0,0,0) which can be determined from the condition G(1) = 1.

```
#'Q15←SIMPLIFY(SUBSTITUTE(RIGHT(Q14),1,'U) = 1)#
```

The machine types out INDETERMINATE, indicating that ∞* 0 has been replaced by UNDEFINED.

```
#EDISPLAY('Q15)#
```

The operator LIMIT will be tried; this operator uses l'Hopital's Rule. It is slow, and so it should not be used when substitution will suffice.

```
#'Q16←LIMIT(RIGHT(Q14), 'U,1)#
#EDISPLAY('Q16)#
```

$$P(1,2,0)=\frac{P(1,1,0)\cdot\left[\theta+2\cdot\theta\cdot(\theta+1)+2\right]}{(3\cdot\theta+2)}$$

$$P(0,0,0)+\frac{2\cdot P(0,0,0)\cdot\theta\cdot\left[\theta+2\cdot\theta\cdot(\theta+1)+2\right]}{(3\cdot\theta+2)}+2\cdot\theta\cdot P(0,0,0)+\mathrm{UNDEFINED}\cdot\frac{1}{4}=0$$

Q16 can be simplified by factoring and expansion.

```
#'Q17←FACTOROUT(EXPAND(Q16),'(1((3*THETA + 2)*(1-2*THETA))))#

#'Q17←LEFT(Q17)*EXPAND(RIGHT(Q17))#

#EDISPLAY('Q17)#
```

Q17 is equal to 1 and can be solved for P(0,0,0).

```
#'Q18←SOLVE(Q17=1, '(P(0,0,0)))#

#EDISPLAY('Q18)#
```

The expression for P(0,0,0) shows that the SOLVE routine could be improved. This expression is now substituted into Q14 in order to produce the final expression for the generating function H.

```
#'Q19←SIMPLIFY(EVALUATE(Q14, Q18))#
```

Taking a census of Q19 shows that it is probably too large to display without being split.

```
#CENSUS(Q18)#

#EPRINT('LAST)#                    1438
```

In the thesis this expression was evaluated numerically to form a table of values. This probem would provide a basis for further work in automatic simplification.

(016)

$$P(0,0,0)+\frac{2\cdot[\theta+2\cdot(\theta+1)\cdot\theta+2]\cdot\theta\cdot P(0,0,0)}{(3\cdot\theta+2)}+2\cdot P(0,0,0)\cdot\theta$$

$$+\frac{\left[\frac{16\cdot[\theta+2\cdot(\theta+1)\cdot\theta+2]\cdot P(0,0,0)\cdot\theta^{2}}{(3\cdot\theta+2)}+16\cdot P(0,0,0)\cdot\theta^{2}-\frac{4\cdot[\theta+2\cdot(\theta+1)\cdot\theta+2]\cdot\theta\cdot P(0,0,0)}{(3\cdot\theta+2)}+4\cdot P(0,0,0)\cdot\theta\right]}{(-2\cdot\theta+1)}$$

(017)

$$\frac{\left[6\cdot\theta^{2}\cdot P(0,0,0)+7\cdot\theta\cdot P(0,0,0)+2\cdot P(0,0,0)+2\cdot\theta^{3}\cdot P(0,0,0)\right]}{(3\cdot\theta+2)\cdot(-2\cdot\theta+1)}$$

(018)

$$P(0,0,0)=\left[\frac{2\cdot\theta^{3}}{\left(-6\cdot\theta^{2}-\theta+2\right)}+\frac{2}{\left(-6\cdot\theta^{2}-\theta+2\right)}+\frac{7\cdot\theta}{\left(-6\cdot\theta^{2}-\theta+2\right)}+\frac{6\cdot\theta^{2}}{\left(-6\cdot\theta^{2}-\theta+2\right)}\right]^{-1}$$

The preceding problem solutions show that the current system can be used for work on existing problems. No one part of the system is particularly weak, but there are many interesting possibilities for improvement.