MASSACHUSETTS INSTITUTE OF TECHNOLOGY
PROJECT MAC

Artificial Intelligence
Memo No. 197 May 1970 *

A Simple Algorithm for Self-Replication

Terry Winograd

*Originally issued as term paper, November 1967.

A recurrent topic of interest in the theory of automata has been the possibility of self-reproducing automata, particularly those which could reproduce globally through an application of a local algorithm. In such a device, the "growth" at any point would depend at any time only on the local environment, but the overall effect would be the reproduction of complex structures.

This paper gives an algorithm of this type (an extension of an algorithm brought to my attention by Professor Fredkin) and examines the conditions under which such replication will occur. The system on which it operates will be defined, and the main theorem on its operation will follow from several simple lemmas.

Consider first a system consisting of an infinite checkerboard with positive integers marked in a finite number of its squares. Choose any prime p greater than the largest of these integers. The algorithm is:

(1) At any time n, place in each square the sum modulo p of all the numbers in squares adjacent to it at time $n-1$.

The sense in which this is local depends on the definition of "adjacent", which will be made precise later. As one of many possible examples, consider all orthogonally and diagonally adjacent squares. Taking the initial pattern of Fig. 1, and choosing p=5, we get at succeeding times the patterns of Fig. 2.

We can formally represent the system as a sequence of functions whose domain is the set of vertices of a directed graph and whose range is the set of residues modulo p. We will let $f_o(x)$ represent any initial distribution of integers, and define $F_{n,f_o}(x)$ as the integer at vertex x at time n produced from the initial distribution f_o. We can also define the set of vertices adjacent to x, $A(x)$, as the set of vertices y from which there is an edge directed to x.

Algorithm (1) can then be defined recursively:

$$(2) \quad F_{0,f_o}(x) = f_o(x)$$

$$(3) \quad F_{n+1,f_o}(x) = \underset{y \in A(x)}{(+)} F_{n,f_o}(y)$$

where (+) represents sum mod p.

Define the function $\delta_y(x)$: $\delta_y(y) = 1$, $\delta_y(z) = 0$, $z \neq y$. Any initial function $f_o(x)$ can be represented as the sum of a finite number of integral multiple of δ-functions, one for each vertex y for which $f_o(y) \neq 0$. Call this set of vertices I. Then

$$(4) \quad f_o(x) \equiv \sum_{y \in I} f_o(y) \; \delta_y(x)$$

<u>Lemma 1</u>: For any $f_o(x)$ and all $n \geqslant 0$,

$$(4) \quad F_{n,f_o}(x) = \underset{y \in I}{(+)} f_o(y) F_{n,\delta y}(x)$$

Proof: The case n = 0 is a result of identities (2) and (4). Using induction on n,

$$F_{n+1,f_o}(x) = \underset{z \in A(x)}{(+)} F_{n,f_o}(z) \qquad \text{By (2)}$$

$$= \underset{z \in A(x)}{(+)} \underset{y \in I}{(+)} f_o(y) F_{n,\delta y}(z) \qquad \text{By the induction hypothesis}$$

$$= \underset{y \in I}{(+)} f_o(y) \underset{z \in A(x)}{(+)} F_{n,\delta y}(z)$$

$$= \underset{y \in I}{(+)} f_o(y) F_{n+1,\delta y}(x) \qquad \text{By (2).}$$

This lemma combines two results. The value at any vertex at any time resulting from an initial distribution of a single integer n is equal to the product (mod p) of n and the value which would have resulted at that vertex from a distribution of a single 1 in place of n. The value resulting from an array of initial values is the sum (mod p) of the values which would have resulted at the same time in separate distributions, each generated by one of the initial integers.

For any two vertices y and x, let $P_n(x,y)$ be the number of distinct paths of length n from y to x.

Lemma 2: For any vertices x,y and any $n \geq 0$;

(5) $F_{n,\delta y}(x) = P_n(y,x) \mod p$

Proof: For $n = 0$ the lemma is trivial, as there is exactly one path of length 0 from y to y and none to any $x \neq y$. Any path of length $n+1$ connecting y to x can be represented uniquely as a path of length n connecting y to a member z of $A(x)$, followed by a final edge connecting z to x. Therefore:

$$P_{n+1}(y,x) = \sum_{z \in A(x)} P_n(y,z)$$

$$= \underset{z \in A(x)}{(+)} F_{n,\delta y}(z) \qquad \text{By the induction}$$

$$= F_{n+1,\delta y}(x) \qquad \text{By (2).}$$

To this point there have been no restrictions on the directed graph. Obviously the number of distinct paths connecting two vertices will be determined by the geometry of the graph. In order to achieve the desired replication properties, we will consider a special class of graphs, which we will call "directional."

Definition: A directional graph is a directed graph whose edges can be partitioned into a finite number of classes such that:

(a) There is exactly one incoming and one outgoing edge of each class at each vertex.

(b) Any two paths beginning at the same vertex and containing the same numbers of edges of each class end at the same vertex.

(c) Two paths, each containing edges all of one class, will have at most one point in common unless both paths are made up of edges of the same class.

The geometric motivation for this definition is clear. If we think of a graph in Euclidean n-space which is invariant under translations carrying vertices into vertices, we can consider the directions of edges as their classes, i.e. all parallel edges will be in the same class. We will call the classes "directions."

The definition does not imply all of the restrictions implied by the intuitive "checkerboard" model. We will list some of the implications and possibilities.

1. The directions are not necessarily independent.

   In the example on page 1, a single edge in a diagonal direction leads to the same point as the combination of a horizontal edge and a vertical edge.

2. The directions are not necessarily symmetrical.

The fact that x is adjacent to y does not imply that y is adjacent to x. If we use a checkerboard and consider only the three squares immediately to the left of x as adjacent to it, the conditions are satisfied.

3. Graphs with the same number of directions are not necessarily isomorphic.

   If we consider the dual graph of a hexagonal tiling of a plane, there are six directions which are not independent. A three-dimensional checkerboard with only orthogonal adjacency has six independent directions.

4. Graphs can contain loops but in at most one direction.

   In any graph, we can consider each vertex connected to itself in a zero-direction. In our origianl example, this would result in a ninth replication located in the position of the original. If there are loops in two different directions, property (c) will be violated.

In general, any filling of Euclidean n-space with identical solids, with any rule of adjacency will produce a directional graph. The adjacency need not be the intuitive one -- it could include such patterns as the knight's move on a chessboard. There are other possible directional graphs which cannot be represented in Euclidean space.

<u>Lemma 3:</u> In a directional graph of m directions, for any integer k 0, $F_{p^k, y}(x)$ is equal to 1 at m points, each of which is connected to y by a path consisting of p^k edges all of the same direction, and is zero at all other points.

Proof: Any path of length n from y to x will contain a definite number of edges of each direction. We can associate with it a vector of integers $(n_1, n_2, \ldots, n_m)$, where $\sum_{i=1}^{m} n_i = n$. The set of all paths of length n from y to x can be partitioned by grouping all paths with the same direction vector D. Property (a) implies that every distinct permutation of the edges indicated by D is actually a path, and property (b) guarantees that it leads to x. Therefore the number of distinct paths in each block of the partition must be equal to the number of permutations of n objects of which $n_1, n_2 \ldots, n_m$ are alike. For $n = p^k$, this will be:

$$\frac{p^k!}{n_1!n_2!\ldots n_m!}$$

It is easily established that this will be divisible by p unless there is some i such that $n_i = n$, $n_j = 0$, $j \neq i$. Property (c) implies that there will be at most one path from y to x meeting this criterion. Therefore the number of paths will be the sum of numbers of which all but at most one are divisible by p, and will therefore by congruent (mod p) to 1 for exactly those cases in which there is a path made up of edges all in the same direction, and to 0 for all other vertices.

Definition: A distribution F(x) of integers on a directional graph is a replica of a distribution G(x) if the vertices of the two can be put into a one-to-one correspondence such that:

a. The integers at corresponding vertices are the same.

b. If x and y are vertices of F(x) corresponding to x' and y' of G(x), and if D is the direction vector for any path connecting x to y, then any path starting at x' and having the same vector ends at y'.

Theorem: Given any directional graph with m directions, and any initial distribution of integers at a finite number of its vertices, then for every prime p greater than the largest of these integers, there exists a k_p such that the result of applying algorithm (1) p^k times for any $k > k_p$ will be m identical replicas of the original distribution, each translated a distance of p^k edges from the original in a single direction.

Proof: Lemma 1 implies that the result at any point is the sum (mod p) of the results produced by the points of the original distribution taken individually. Lemma 3 shows that these individual images are the indicated translations of the indicated values. Property (c) of directed graphs implies that no two images of the same point can coincide at any time, and that the image of a point y in direction A and the image of point x in direction B can coincide at most once. Since there are a finite number of original points and a finite number of directions, there is a time k_p after which no two images can coincide, and the properties of the replica are assured by the properties of a directed graph.

It is also interesting to note that there exists a k'_p such that for all $k' > k'_p$ the result of applying the algorithm $p^{k'} + p^k$ times (where k is greater than k_p but limited to some small value depending on k') will be m^2 replicas. Essentially, the $p^{k'}$ repetitions produce m widely spaced replicas, and the p^k additional applications of the algorithm produce m replicas of each of these. Similarly for an infinite number of appropriate values of k, the result of applying the algorithm

$\sum_{i=1}^{n} p^{k_i}$ times will be m^n identical replicas of the original pattern.

Figure 1-2

1	2	3	n = 0 (Figure 1)
		4	

1	3	1	0	3	
1	2	3	1	2	n = 1
1	3	0	0	2	
		4	4	4	

1	4	0	4	4	3	3	
2	1	3	4	0	3	0	
3	0	3	0	1	1	2	n = 2
2	1	1	2	0	1	3	
1	4	3	2	0	1	1	
		4	3	2	3	4	

1	0	0	3	3	1	0	1	3	
3	2	2	1	1	4	0	1	3	
1	4	1	1	0	0	4	2	0	
2	1	1	2	4	1	0	3	0	n = 3
1	4	2	0	0	3	4	1	1	
3	2	2	2	0	4	4	2	3	
1	0	2	2	1	4	3	0	0	
		4	2	4	3	4	2	4	

1	1	1	3	1	2	4	2	4	4	3	
4	0	3	3	2	0	3	2	2	0	1	
0	3	2	2	1	3	4	3	3	2	1	
1	2	3	4	2	4	4	1	1	4	3	
4	1	1	0	4	2	0	3	2	2	1	n = 4
1	2	4	4	3	1	2	3	2	4	4	
0	3	0	4	4	1	4	1	1	4	4	
4	0	4	4	2	1	3	3	2	1	2	
1	1	2	1	3	2	1	2	1	2	4	
		4	1	0	4	1	4	0	1	4	

continued

1	2	3	0	0	1	2	3	0	0	1	2	3
0	0	4	0	0	0	0	4	0	0	0	0	4
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	3	0	0	0	0	0	0	0	1	2	3
0	0	4	0	0	0	0	0	0	0	0	0	4
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	3	0	0	1	2	3	0	0	1	2	3
0	0	4	0	0	0	0	4	0	0	0	0	4

n = 5