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THE ART OF SNARING DRAGONS

Harvey A. Cohen

ABSTRACT

DRAGONS are formidable problems in elementary mechanics not amenable to solution by naive formula cranking. What is the intellectual weaponry one needs to snare a Dragon? To snare a Dragon one brings to mind an heuristic frame - a specifically structured association of problem solving ideas. Data on the anatomy of heuristic frames - just how and what ideas are linked together - has been obtained from the protocols of many attacks on Dragons by students and physicists. In this paper various heuristic frames are delineated by detailing how they motivate attacks on two particular Dragons, Milko and Jugglo, from the writer's compilation. This model of the evolution of problem solving skills has also been applied to the interpretation of the intellectual growth of children, and in an Appendix we use it to give a cogent interpretation for the protocols of Piagetian "Conservation" experiments. The model provides a sorely needed theoretical framework to discuss teaching strategems calculated to promote problem solving skills.

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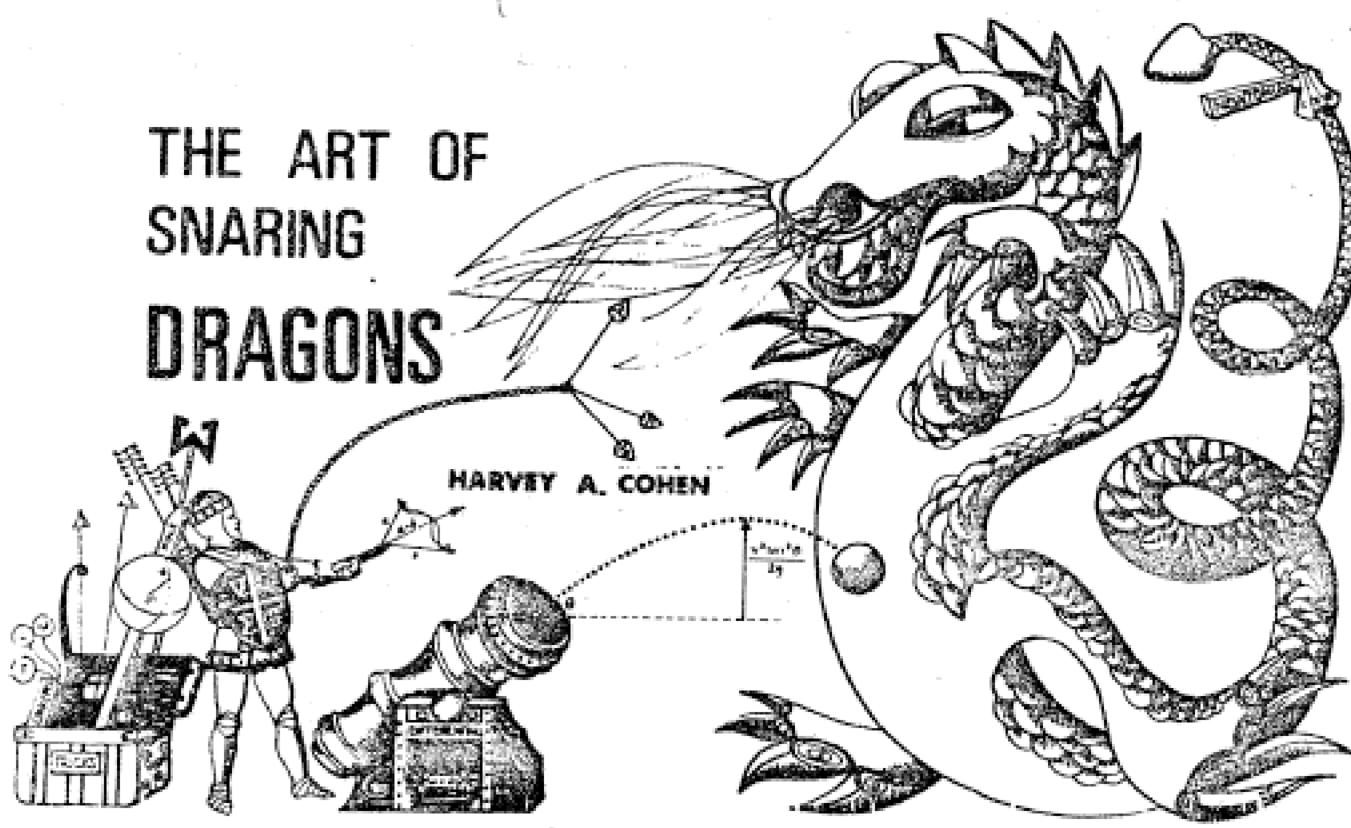
HARVEY A. COHEN



1. INTRODUCTION

Every teacher of physics would, I feel, assert that he strives to teach students not only how to solve certain paradigm examples -- but that he also hopes to impart a cluster of generalized skills in problem solving that will equip students to comprehend and analyse a greater range of problems than could possibly be discussed in lecture classes and tutorials. But is there any specific way to promote this objective ? The purpose of this introduction is to recount some of the more accessible ideas in a teaching strategem I have been developing for this purpose. This stratagem shares elements in common with what I call the Fermi stratagem (in Physics teaching) and Polya stratagem (in Mathematics teaching).

In the Fermi stratagem students are posed problems of a more project-like character. Some such Fermi problems are relatively open-ended (e.g., "How, in terms of physics, do we walk and run?"). Other Fermi problems have a definite solution but are of a "non-standard" form requiring the skillful selection and artful utilization of perhaps quite elementary physical models. Teachers wishing to follow the Fermi stratagem face two difficulties. The first is the scarcity of Fermi problems, or rather the scarcity of compilations of such problems. In this regard Walker's "Flying Circus"[1] is a very welcome addition to the Physics teaching literature. This writer has also compiled a collection of non-standard problems (which he calls "Dragons") in elementary mechanics, "A Dragon Hunter's Box"[2] of which some of the Dragons may be aptly characterized as Fermi problems. The second difficulty in introducing Fermi problems is the absence of any comprehensive tutor's guide embodying theoretical analysis and practical experience in the presentation and effective utilization of such problems. In fact the casual introduction of Fermi problems into certain innovatory courses of recent years has often lead to obvious failure, as the students participating have lacked any model of how to proceed in tackling any problem other than those more conventional problems which I term "formula crankers."

In the Polya stratagem, students already familiar with the tricks and techniques needed for particular problems, are given specific instruction in powerful general problem solving ideas - - what Polya terms mathematical heuristics. The style of presentation, as evidenced by the structure of "Mathematics and Plausible Reasoning"[3] is to first explain a particular heuristic, and demonstrate its applicability to a particular problem: the student is then posed a graded set of problems which are amenable to solution via that heuristic. The writer is not aware of any extensive application of the Polya stratagem to physics teaching.

My own teaching stratagem grew out of an attempt to implement the key ideas of Fermi and Polya in the context of a college course in elementary mechanics. I was especially keen to get away from the traditional emphasis on problems which may be characterised as "formula-crankers" and to engage students in problems which had more of the flavour of research problems in physics, such as Fermi problems. There are in fact very few published Fermi problems in elementary mechanics, and it seems the typical problem actually posed by Fermi was "How many piano tuners are there in New York?".[4] So in order to produce a significant compilation of challenging problems for student use I was obliged to devise a number of new problems in elementary mechanics which I termed Dragons to express their formidable character. In line with this playful terminology, the first compilation of Dragons was produced in a hand lettered and illustrated booklet[5] entitled "What G Killed Ned Kelly? and Other Problematical Dragons"(Ned Kelly - - an Australian folk hero - - the last of the bushrangers - - was hung in Melbourne in 1867). The "Ned Kelly" Dragon book was used in conjunction with the lectures and tutorials of a course in elementary mechanics, AM204 Second Year Mechanics at La Trobe University, in Melbourne, Australia during 1972.[10]

It is now opportune to discuss the pedagogic principles underlying the selection and construction of the Dragons of the original compilation[5] and its successor[2]. The Dragons were conceived as providing scope for the discussion of problem solving per se rather than particular physical principles. An underlying assumption was that many students try to solve problems in accord with the following model:

<u>The Formula Cranker's Model</u>

Step 1. Look at the problem solved, P.

Step 2. Scan one's reportoire of all the problems one can solve, until one finds S similar to P.

Step 3. Apply the algorithm used to solve S, to P.

I've called this model the Formula Cranker's Model of Problem Solving as this model will, in fact, be of some real service to a student in the solution of a formula cranker - - a problem in which has been specified formally precisely those elements to be substituted in a familiar formula: for instance if shape parameters (such as might be involved in a moment of inertia) are not explicitly labeled and specified the "similar" problem must not devolve on such parameters. My Dragons were selected or constructed so that like the real problems tackled in research the Formula Cranker's Model would fail. Consider first the Dragon MILKO of Fig. (iv). Because MILKO explicitly seeks the determination of the pressure at the bottom of a cylinder-like volume (the interior of a milk bottle), this Dragon is clearly "similar" to the calculation of the pressure at the base of a cylindrical column of liquid. In this sense the Dragon is also "similar" to other calculations of base pressure upon the sole of one's shoes. Hence applying to MILKO the algorithm of the "similar" problems, the base pressure P is given in terms of the base area A and the total weight of the contents of the bottle, W, as

$$p = W/A$$

This expression is entirely false, and is an instance of how the Formula Cranking Model can lead to an inappropriate formula. Consider next the Dragon JUGGLO of Fig. (xiv). It happens that this Dragon may be successfully snared using the same formulas as are applied to the calculations of the mechanics of a rigid body. Yet as juggling is in no sense "similar" to a rigid body, students following the Formula Cranker's Model of action will not arrive at such an analysis (as is given under the caption "In Toto" in Section 3). That is, by this example, we see how the Formula Cranker's Model may prevent students from recognizing the applicability of quite familiar algorithms. The third point to be made about the Formula Cranker's Model is that even if that following this model one determined an appropriate algorithm, application to the given problem may lead to a mess of algebra which is hard to untangle to finally solve the problem. An illustration of this sort of phenomena is provided in Section 2, below the heading Formula Crank.

The above examples indicate that exposure to those formidable (yet elementary) problems I've termed Dragons highlights to students the inadequacy of the Formula Cranker's Model of Problem Solving. But in fact this is only a minor aspect of what can be learnt from such encounters. Particularly when one has in fact produced the canonical wrong answer to a Dragon, a study of such encounters, using introspection and observation of other students, reveals the sort of mental construct -- collection of associated ideas -- one has brought to bear on the problem.

How in fact does one solve physics problems ? Over the past few years I've listened intently to many attempts by students and physicists to snare the Dragons of my collection.[2] These observations (protocols is the jargon word in psychology) support the contention that in solving such problems one uses a structured collection of associated ideas that I've termed a heuristic frame. There appears to be only a relatively small number of heuristic frames available to any individual, of the order of twenty. In Table 1, the anatomy of a heuristic frame is revealed.

TABLE 1

THE ANATOMY OF A <u>HEURISTIC</u> FRAME

COMPONENT	DESCRIPTION
(Core) heuristic	An elemental, crude problem solving idea, probably acquired in childhood.
Problem Reduction Devices and Algorithm Selector	How to reshape the problem and which algorithm to apply.
Debug routines	What to do when things "go wrong".
Demons: Warnings, Caveats, Flags, Pointers	Miscellaneous: "Watch out" "Try another heuristic frame"

In Table 1 and elsewhere in this paper, by an algorithm is meant a highly specific procedure or formula. The (core) heuristic of a heuristic frame is the same sort of mental object as what Polya[3] termed a heuristic -- a problem solving idea of some potency. (Polya confined his attention to mathematics, however). Problem reduction involves putting the problem in a form suitable for the application of particular algorithms. If the unexpected happens -- or even when one is informed that the answer derived is "wrong" -- one calls upon the Debug Routines of the heuristic frame. Also linked with the other components of a heuristic frame are what I've termed Demons: the image is of some little beast that waits for some specific little occurrence to trigger his attention -- when he passes on his message. At any rate, under the heading of "Demons" are lumped together some miscellaneous ideas bound in the frame, such as warnings, caveats, and directives to other frames. A few examples of Demons are presented later in this paper.

The concept of Heuristic Frames provides a description of the evolution of problem solving skills in terms of

a) The growth in one's reportoire of algorithms.

b) The elaboration and augmentation of the components of one's heuristic frames.

The latter process is termed the 'debugging of heuristics': in debugging the core heuristic is essentially unalterable, only the other components of the frame can be edited. A simple description of problem solving in terms of the components of heuristic frames is contained in a model which is called the Horse and Cart or H.A.C. Model (of problem solving)

TABLE II

HORSE AND CART MODEL OF PROBLEM SOLVING (H.A.C.)

TO H.A.C.

Step 1.	Given a problem, choose a Heuristic
Step 2.	Reformulate the problem and select an Algorithm
Step 3.	CRANK the algorithm
Step 4.	In case of trouble, DEBUG.

The H.A.C. Model is presented in Table II. This model essentially states that the choice of Heuristic precedes the choice of an Algorithm that does the actual Cranking of a problem. As stated above, the model is over simple, but has proved to be an effective tool in promoting problem solving skill, by providing a descriptive basis for self-assessment and student counselling. Thus in total, this paper deals with a teaching stratagem based on two models:

i) A model for intellectual development in terms of the debugging of heuristics

ii) A model for problem solving.

An example of how a tutor may aid the intellectual development of a student by directing attention to the debugging of one particular heuristic is provided by the following example taken from my tutorial records.

A student complained that he didn't "understand" gyroscopic effects. What that meant was that he could follow the mathematical presentation given in class, yet the behaviour was still surprising. I probed further and found that if a flywheel was spinning in a vertical plane, and a torque about the vertical axis was applied for an instant, this student expected the fly-wheel to remain vertical, but for its plane to rotate about the vertical axis.

Fig. (i)

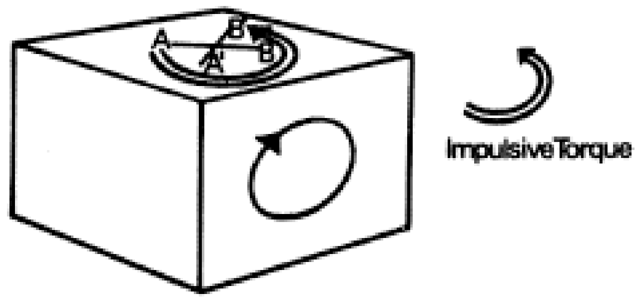


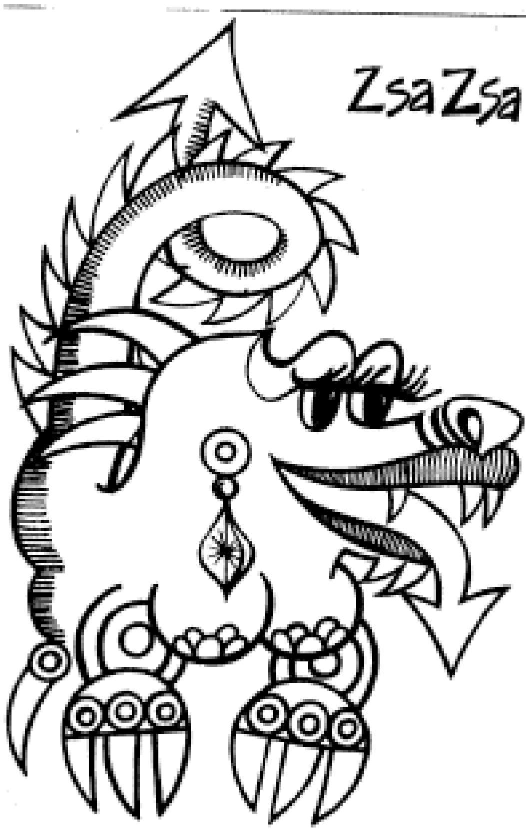
Figure (i) Imagine that a spinning flywheel is placed inside the box (drawn here in isometric projection) with the plane of the flywheel parallel to the front face of the box. The spin sense of the flywheel is marked on the front face, and the projection of the wheel, the line AB, on the top of the box. A torque, applied briefly, is indicated by its tendency to twist in the top (horizontal plane), rather than as a vertical vector. One common student expectation is that the new position of the flywheel has the projection A'B' on the top of the box, corresponding to a rotation of the plane of the flywheel about the vertical.

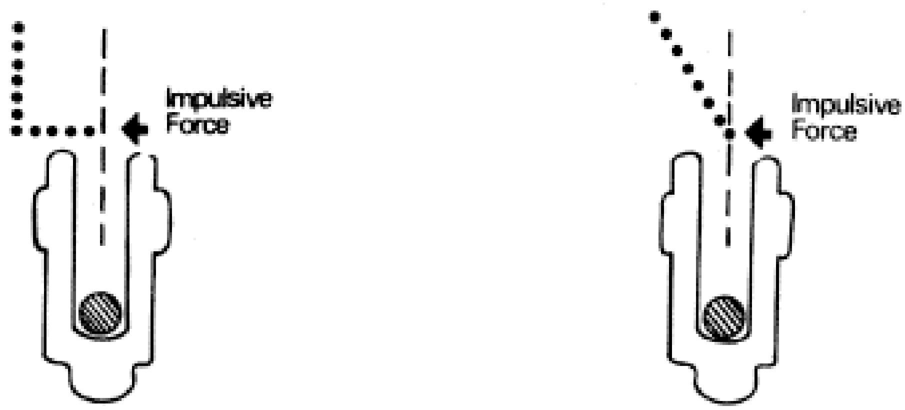
Figure (i) is the diagram that was drawn while endeavouring to clarify the students expectation. It is clear that the student was here invoking a <u>heuristic</u> "Parallel" -- the idea that the "effect" of a "force" is a displacement in the "direction" of that "force". (The direction in this specific case is a screw sense). The student had selected an algorithm which could be formally stated as

Twisting Force X Time = Amount of Twist

This particular algorithm is appropriate to a high friction environment such as the domestic arena of a young child. It is essentially an Aristotlean algorithm - part of a physics where forces "cause" displacements in velocity. In order to help this student debug I constructed an argument involving the same <u>heuristic</u> (Parallel) and patently presenting a choice between Newtonian and Aristotlean algorithms for forces:

Consider a cannon firing at a target (drawn a schematically from above in Fig.(ii)





Aristotlean Algorithm Newtonian Algorithm

Fig (ii) Dashed line is the unperturbed trajectory of a cannon ball. Dotted line denotes new trajectory after application of an impulsive force according to (A) Aristotlean Algorithm, (B) Newtonian Algorithm.

Suppose just as the cannon ball emerges from the barrel it is given a short sharp knock. Then, in accordance with the expectation portrayed in fig. of generalized impulsive forces causing a spatial displacement in the direction of application the ball should be deviated as shown in fig. (ii)A. Now of course what actually would take place is properly demonstrated in fig. (ii)B - - the effect of the impulsive force is to give the ball a transverse component of momentum to determine the subsequent trajectory of the ball. Returning to the flywheel problem, it likewise follows in formal terms that the effect of an impulsive torque about the vertical is to produce angular momentum about the vertical, which has to be compounded with that pre-existing.

This point is well made by a drawing such as Fig. (iii).

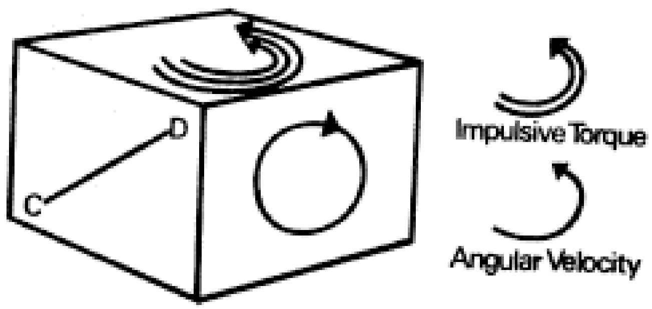


Fig. (iii)

In this figure the original and the additional (angular) momenta are shown as screw senses on the sides of a box containing the flywheel. But these two screw senses - - compounded in a Newtonian way (algorithm) - - must be just the projection of the resultant motion of the flywheel. So - - imagining arrows drawn on the flywheel showing rotation sense - - one deduces that the flywheel - - having suffered the impulsive torque (double arrows in the figure) - - changes its plane of motion: the new projection of the flywheel is shown in fig (iii) as the straight line CD.

In summary, my first concern as a tutor was to aid this student in debugging the <u>heuristic</u> (Parallel) he had sought to invoke for the processional problem. (Compare computer programing: one has to debug the programs one actually writes; on the other hand it pays to learn of other programs). Confronted with this student a tutor espcusing a different strategy might have replied: "Don't look at a wheel like that. Look at a wheel as composed of little parts,[2] and consider the effect of the applied forces on each little part . . ." This particular approach invokes the <u>heuristic</u>

"Divide and Conquer" (discussed later in this paper) and it is well for a student to see a "Divide and Conquer" approach to a tantalising problem: however, to repeat, in line with the above described model for problem solving attention to the debugging of a heuristic is paramount, and would be a tutor's first concern.

Physics problems depend on a small number of heuristics specific to physics. In this paper we are to discuss just seven of these heuristics:

Formula Crank

To Paradigm

In Toto

Fibre/Capillary

Add Effects (and Subtract Effects)

Divide and Conquer

Process

In this list "Formula Crank" is none other than to apply the Formula Cranker's Model of problem solving, the other heuristics are described in Section 2. For the moment it is important to note just how few there are, and that in my teaching stratagem, explicit names are given to each heuristic. Now in the Polya strategem students gain "familiarity" with a particular heuristic by applying that heuristic to a range of different problems. In my stratagem this is also done, but much stress is laid on applying different heuristics to the same problem -- to stimulate the debugging of these heuristics. And also to overcome what I call Magic Key Thinking -- the idea that there is just one way of looking at a given problem (a unique heuristic)

Just what are these heuristics, and how good are they in practice? Section 2 is devoted to delineating these six heuristics, and showing their application to the snaring of the Dragon Milko of Fig. (iv), Section 3 shows how four of these heuristics motivate algorithms that successfully snare the Dragon "Jugglo" of Fig. (xvi). This discussion of Section 2 and 3 will prove of value to any teacher who wishes to discuss the two Dragons, Milko and Jugglo with students -- using the tutorials as heuristics debugging scenes where the tutor is equipped to guide an illformed but not heuristically misguided student foray at these Dragons. In Section 4 the teaching stratagem presented here is reviewed. The Appendix shows the application of the theoretical framework of this paper to aspects of the intellectual development of children. The "debugging of a heuristic" is thereby demonstrated in a simple setting, various heuristic morals are drawn.

2. MILKO

Preliminary Remarks

The problematical Dragon "Milko" of Fig.(iv) is reproduced from my compilation "A Dragon Hunter's Box". Please read the first paragraph of this Dragon. I have posed this problem to many undergraduates, graduates, engineers and professional physicists. Invariably they jumped to the conclusion p = p'. When informed that this was the canonical wrong answer, a line of argument often developed which made plain the heuristics invoked, and the debug routines, caveats, and warnings that were associated with particular heuristics. The later paragraphs of this Dragon contain a measure of suggestion and counter-suggestion designed to provoke such an analysis by the reader.

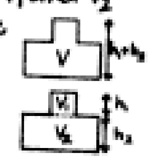
So . . . what heuristics are there for snaring "Milko, and just how is it done?

★ A milk bottle is allowed to stand so that the cream rises to the top: this occurs without any change in total volume. Does the pressure near the base of the bottle change?

★ Surely, since the total weight W of the liquid remains constant, the pressure on the base, of area A, being W/A remains constant.

★ After separation, the pressure immediately below the cream is less than what it was at that level before separation. So maybe ...

★ A mathematician uses a regular milk bottle of volume V, containing liquid of density ρ. After settling time, the liquid separates into two components of density ρ_1 and ρ_2, which occupy volume V_1 and V_2 respectively, as indicated, where $V_1 = h_1 A_1$, $V_2 = h_2 A_2$. So he calculates a change in pressure after separation.



★ A would-be physicist challenges the mathematician: "There are no different bulk forces acting after separation; the pressure is unchanged."

★ The mathematician deduces: the w.b.p. takes his milk in cartons.

Fig (iv) Reproduced with permission from H.A. Cohen "A Dragon Hunter's Box", Hanging Lake Books, Warrandyte, Victoria, Australia (1974).

In Toto

The heuristic "In Toto" embodies treating the diverse parts of a physical system as a single system. In the text of "Milko" the statement of the "would-be physicist" suggests that the w.b.p. - - like may first exposed to this Dragon - - has adopted an "In Toto" viewpoint and applied an elementary statics algorithm to equate the total gravitational force W to the product of base area A base pressure.

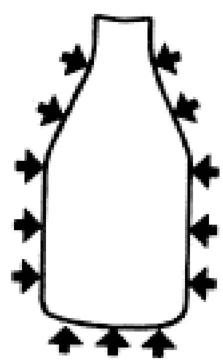
On being informed that they have given the canonical wrong answer for "Milko," "In Toto" champions - - who have treated the milk as a whole - - tend to

a) Check whether they have included to much in the whole
b) Check whether they have included too little in the whole
c) Switch to "Divide and Conquer" viewpoint.

The routines (a) and (b) are debug routines (or part of debug routines) associated with the "In Toto" heuristic. (c) is what I'd simply call a flag, or pointer to an alternative heuristic. Of course the more skillful problem solvers are more effective in invoking the above (and other) debug routines.

Debug routine (a) suggests to check what was included in the quantity W: and clearly it was the weight of the bottle, so that W/A is the pressure at the base of the bottle at the glass/table boundary. At this stage there's a strong inducement to switch to "Divide and Conquer" and check whether the pressures above and below the glass base of the bottle are equal or not. (See the discussion under the heading "Divide and Conquer".)

Debug routine (b) leads to the question "Is the milk really just sitting there with just the force of gravity and base pressure (times base area) holding it in place?" This leads to the more particular question as to whether the sidewall pressure forces can have a net vertical sum. Now sidewall pressure forces don't cancel - - at least they do where the walls are vertical - - but not where the bottle walls are slanting. As indicated in fig (v) the reaction forces have a net downward sum X when the contents are homogeneous, X' after separation of cream.



Fig(v) Sketch of wall reaction (pressure) forces acting on the contents of a milk bottle.

By the usual statics algorithm,

$$pA = W + X \quad , \quad p'A = W + X'$$

When the milk separates, the density of liquid in the neck is less, so that pressures in this region are less, so that the sum of all the sidewall reaction force is less after separation,

$$X' < X$$

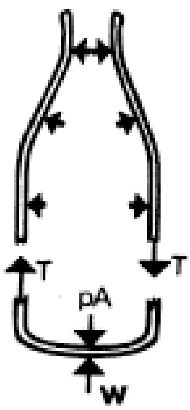
and hence the conclusion $p' < p$.

Divide and Conquer

The Heuristic that I've called "Divide and Conquer" exhorts one to divide a physical system into a number of parts, and to solve the various sub-problems before assembling the component parts and the corresponding sub-problems. The example of this heuristic applied to snare JUGGLO is rather more cogent than what we do here.

We take as starting point the calculation of pressure below the base of the bottle presented above (See In Toto). Break the bottle into the parts shown in fig (vi).

Fig (vi)



The vertical tension in the sidewalls of the bottle is easily overlooked. By considering the equilibrium of the base one deduces

$$W = pA - T \;, \qquad W = p'A - T'$$

where T, T' are the (corresponding) vertical sums of the side-wall forces at the base. By considering the equilibrium of the sides of the bottle, one deduces that T (T') are exactly cancelled by the vertical sum of the forces due to liquid pressure acting on the sides, i.e.

$$T = X \;, \qquad T' = X'$$

where X(X') is the same quantity as determined in the "In Toto" discussion. Thence, on comparing X and X', one deduces that base pressure is less after separation, $p' < p$.

Formula Crank

The heuristic "Formula Crank" involves the application of what in the introduction was called the Formula Cranker's Model of Problem Solving. To illustrate the potency of "Formula Crank" -- I will repeat an apocryptal story about Feynman and his early work.[15] It appears that in a discussion Jauch informed Feynman of the 1931 paper of Dirac which showed that there was an analogy between unitary transformations in quantum mechanics and the exponential of S where S was a classical quantity. Whereupon there and then Feynman proceeded to manipulate the "analogous" classical expressions as though they were the quantum mechanical unitary transformations, yield a first crude version of what was to become his important "Space Time Formulation of Quantum Mechanics". Clearly this was "Formula Crank" motivated work -- but Feynman had to call upon all his intellectual resources -- his elaborated (debugged) heuristics -- to make a mass of meaningless formulae into an important element of modern physics. To illustrate the impotency of "Formula Crank" by itself -- here is how it might be applied to MILKO. First to recapitulate the discussion of the Introduction. A Formula Cranker will take recourse to other calculation of base pressure, as of the pressure at base of one's shoes, to calculate a constant base pressure

$$p = W/A$$

in terms of the weight of contents of milk bottle and base area. If the validity of this result were queried, what could a Formula Cranker do? Very little, observation suggests. The weakness in Formula

Crank" is that there is no means to debug a solution other than relatively capriciously selecting a new algorithm. So as a next step, consider the application of what might be billed as the most comprehensive algorithm for calculating pressures, the formula

$$p = \sum_i (\rho\ g\ h)_i$$

where the summation is other layers of length h_i of material of density ρ_i. We apply this formula to the simplified shape "mathematical milk bottle". For homogenous milk base pressure is

$$p' = h_1\rho_1 g + h_2\rho_2 g$$

This algorithm isn't enough. Conservation Algorithm yields

$$\rho_1 V_1 + \rho_2 V_2 = \rho(V_1 + V_2)$$

where by geometry these volumes are given in terms of areas and heights by

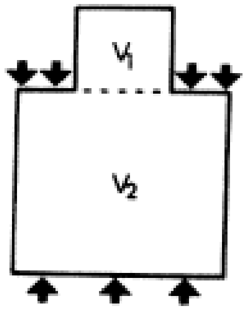
$$V_1 = A_1 h_1,\ V_2 = A_2 h_2$$

Whence such formulae as

$$\frac{p'}{p} = \frac{(V_1 + V_2)\ (\rho_1 h_1 + \rho_2 h_2)}{(h_1 + h_2)\ (\rho_1 V_1 + \rho_2 V_2)}$$

From this formula it is clear that p' is $\neq p$, but it takes a measure of careful algebraic manipulation before the barest qualitative features emerge. In contrast, consider an "In Toto" motivated attack. See Fig. (vii), in which the arrows indicate the vertical forces acting on the contents of the "regular" milk bottle of the mathematician.

Fig (vii)



For homogenous milk

$$pA_2 = (V_1 + V_2)\rho g + (A_2 - A_1)h_1\rho g$$
$$= W + (A_2 - A_1)h_1\rho g$$

For stratified milk, cream in volume V_1, "water in volume V_2

$$p'A_2 = W + (A_2 - A_1)\ h_1\rho_1 g$$

In this case, as cream is lighter than milk, i.e. $\rho_1 < \rho$, it follows that $p' < p$. The point being made is that in an argument motivated by the heuristic "In Toto", the Algorithm gets marshalled - - is interpretable and therefore under control A Formula Cranker needs mathematical skills of high order to organize an elementary physical calculation.

Columns (Reduction Device A)

The heuristic "Fibre" is a valuable problem solving idea utilised by Galileo in his "Dialogues Concerning Two New Sciences".[16] Galileo imagined a solid beam to be composed of parallel fibres, or filaments, effectively independent, the total tensile load carried by the beam being the sum of the tensions in each filament. What must be stressed is that although Galileo talked in terms of beams, which often are made of fibrous material (wood), his discussion was intended to apply to beams of any solid material, so that the fibres are truly fictions. In fact Galileo mentioned stone beams in his discussion. Galileo used "Fibre" skillfully and was probably aware of such caveats to be attached tothis heuristic as that one much check that it's a reasonable first approximation to consider the fibres independent.

The <u>heuristic</u> "Column" is very closely related to "Fibre". One might say it is merely "Fibre" applied to fluids, so that it is the very same heuristic. "Column" suggests that one analyses pressure differences in fluids by considering the body of the fluid to be made up of cylindrical columns. The caveat of non-interference between adjacent fibres/columns is still relevant here. In the next sub-section we will discuss further the issue as to whether Fibre/Column are two <u>heuristics</u> or one. For the moment, lets consider a particular column approach to Milko. We'll present not only a successful solution route along which "Columns" will pull a Statics Algorithm -- but we'll also note one of the cul de sacs.

Consider the two fluid columns shown in Fig.(vii), one near the axis and the other well off the axis. This suggests a bug-- it appears at first that the pressure must differ along the base of the milk bottle -- as the two columns are of different height. However this bug arose by ignoring wall pressure. By considering the static equilibrium of a horizontal fibre (column) of fluid

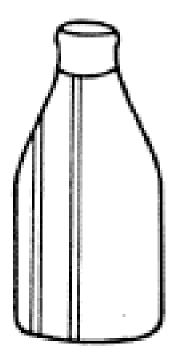


Fig (viii)

it is possible to convince oneself that in fact there is a unique base pressure. It remains much easier then to consider a column about the axis of the bottle. To calculate the base pressure, there are two cases: a) Contents homogenous milk: pressure p

b) Contents stratified: pressure p'

Consider central columns, on base area δA, in the two cases, The density of contents of column (b) is less than that of column (a) -- as basically (b) has an excess of cream. Expressing this evaluation in terms of weight,

$$p\delta A > p'\delta A$$

$$\text{or} \quad p > p'$$

That is, the base pressure decreases after separation. At this stage of the calculation, one might return to examine the fine detail re the two columns a and b to realise that we have ignored side forces: no matter if sides are vertical as these forces didn't contribute to the sums considered. In fact the prime heuristic message to be learned from this calculation could be summed up in the following heuristic:

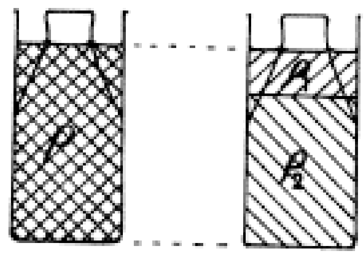
A: "Select a thin vertical column that does not intersect any sidewalls"

A is one of the Problem Reduction/Algorithm Selection Devices associated with the "Column" heuristic.

Columns (Reduction Device B)

We'v already suggested that the preceding application of Columns amounted to an application to a hydrostatic context of the "Fiber" heuristic. However, the special convenience of the central column of Fig. (viii) is not its thinness, but that having vertical sides, the thrusts on the walls of the column had no vertical components. So that it's natural to consider columns of very large cross-sectional area in hydrostatics. All will go well, unless the column hits a slanting wall. This is a bit of a nuisance (bug), but there is a way out as detailed below. But in debugging "columns" to motivate a solution like that presented below -- the connection with "Fiber" is getting a little remote. Thus one should say that originally "Column" was just a portion of the heuristic "Fiber" but ultimately, with elaboration (debugging) it assumes autonomy as an independent heuristic -- possessing a core common with Fiber. This is a very important process in intellectual development that I term replication of heuristics: the mother heuristic spawns a daughter with many common elements. However, the idea of replication is part of my more elaborate psychological model of problem solving -- and its presentation I do not see as part of the teaching stragem I espouse. Certainly if the sort of application of Columns presented below is as far as this heuristic is elaborated, the solution given is still reasonably conceived as motivated by "Fiber" debugged for hydrostatics.

Fig (ix)



Let us make the thought experiment of enclosing the milk bottle in a cylinder, sharing the same base, as in Fig.(ix). Our aim is to reduce the MILKO problem to a discussion of the pressure at the base of columns standing on the base of the milk bottle. Suppose that in the case when the milk bottle contains homogenous the space outside the bottle, but inside the cylinder is filled with milk to the same level as within the bottle: the volume of milk exterior to the bottle we call V_{ext}. Likewise, in the case when the milk has separated into components of density ρ_1(cream), and ρ_2 (creamless milk), suppose the exterior volume V_{ext} within the cylinder is filled to corresponding levels with cream and creamless milk (see Fig.(ix)). The presence of the bottle stands in the way of a "Columns" motivated algorithm, but we can justify ignoring its presence. Since pressure depends on depth alone, the pressure on each side of the bottle is the same, so that the pressure at the base of the bottle, p (for homogenous milk) p' (for stratified (separated) milk)) is unaffected if one removes the bottle walls, but leaves the fluid contents just as they were. Then considering the static equilibrium of the <u>columns</u> standing on the base area A of the bottle one has

$$pA = W + \rho V_{ext} g$$

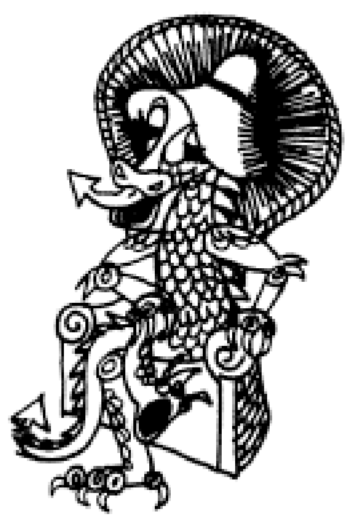
$$p'A = W + \rho_1 V_{ext} g$$

In these equations W is the weight of the contents of the bottle, $\rho_1 V_{ext} g$ is the weight of the fluid in the exterior volume, this fluid being predominantly cream. Hence we see at once that

$$p > p'$$

In summary, the significant driving motive in producing the above derivation is the "Column" heuristic -- relentlessly applied to enable consideration of a vertical column of fluid standing on the bottle base area. This is a striking example of more sophisticated problem reduction: bringing to light a Problem Reduction/Algorith Selection Device which we denote by B, which is roughly as follows:

B: "Choose a vertical column with an "interesting" base. Remove intersecting walls whilst retaining fluid equilibrium"



"Add Effects"

The heuristic "Add Effects" encapsulates the idea of (independent) causes having an additive cumulative effect. A verbal formulation of this problem solving schemata would be:

"If X causes effect E,
and Y causes effect F,
then X + Y causes effect E + F."

To implement "Add Effects" in a given problematical situation one must devise or select quantities that can meaningfully be added together [17]. In fact one aspect of the evolution of the field concept, and vector and tensor notation, of classical electromagnetic theory was the devising of a formalism in which "Add Effects" was more or less "built-in," as is especially exemplified by the "principle of superimposition" for fields. Likewise "Add Effects" is explicit in various additivity rules and implicit in the formalism of all those theories of physics characterised as linear. It is an enlightening struggle to make an "Add Effects" foray at the Dragon MILKO.

In "Layman's Physics" it's the cream and milk minus cream (which we glibly term water) which "cause" the pressure at the base of a milk bottle. A little more formally, if the effect is additive, one would write

$$p = p_{cream} + p_{water}$$

and a like expression for the base pressure after separation, p'. Now the total amount of cream is unchanged after separation so that if quantity alone determines pressure, then

$$p'_{cream} = p_{cream} \quad \text{(false!)}$$

and likewise

$$p'_{water} = p_{water} \quad \text{(false!)}$$

leading to the canonical wrong answer, p' = p. A more recondite, and equally false, version of this argument recalls that the total pressure of a gas mixture is the sum of the partial pressures of the components, so that on (mis)treating the components of milk as gases, one deduces a strict additivity of effect as above.

The obvious bug in the above discussion is that distribution must be taken into account. For the moment, we simplify the discussion by only dealing with the regular shaped "mathematicians' milk bottle." Then in accord with "Add Effects" one envisages milk as the superposition of cream of density $\rho_1 V_1 (V_1 + V_2)^{-1}$ and of milk minus cream = water, of density $\rho_2 V_2 (V_1 + V_2)^{-1}$, both cream and water being dispersed throughout the total volume $V_1 + V_2$. Then

$$p_{cream} = \rho_1 V_1 (V_1 + V_2)^{-1} (h_1 + h_2).g$$

$$p_{water} = \rho_2 V_2 (V_1 + V_2)^{-1} (h_1 + h_2).g$$

So that by "Add effects"

$$p = (\rho_1 V_1 + \rho_2 V_2)(V_1 + V_2)^{-1} (h_1 + h_2)\ g$$

$$= \rho (h_1 + h_2)\ g\ .$$

The <u>heuristic</u> has worked beautifully for milk. However, when we turn to calculate via "Add Effects" the base pressure after separation, we run into that super bug mention in Section 1. To implement "Add Effects" one needs to imagine that (as is shown in fig (x), a "pressure ether" of zero density fills up empty spaces, and transmits pressures so that one can calculate the new component pressures as:

$$p'_{cream} = \rho_1 h_1 \sigma$$

$$p'_{water} = \rho_2 h_2 \sigma .$$

Thus

$$p_{cream} - p'_{cream} = -\rho_1 (V_1 + V_2)^{-1} (V_2 h_1 - V_1 h_2)$$

$$p_{milk} - p'_{milk} = \rho_2 (V_1 + V_2)^{-1} (V_2 h_1 - V_1 h_2).$$

If $V_1 = h_1 A_1,\ V_2 = h_2 A_2$ where $A_1 < A_2$

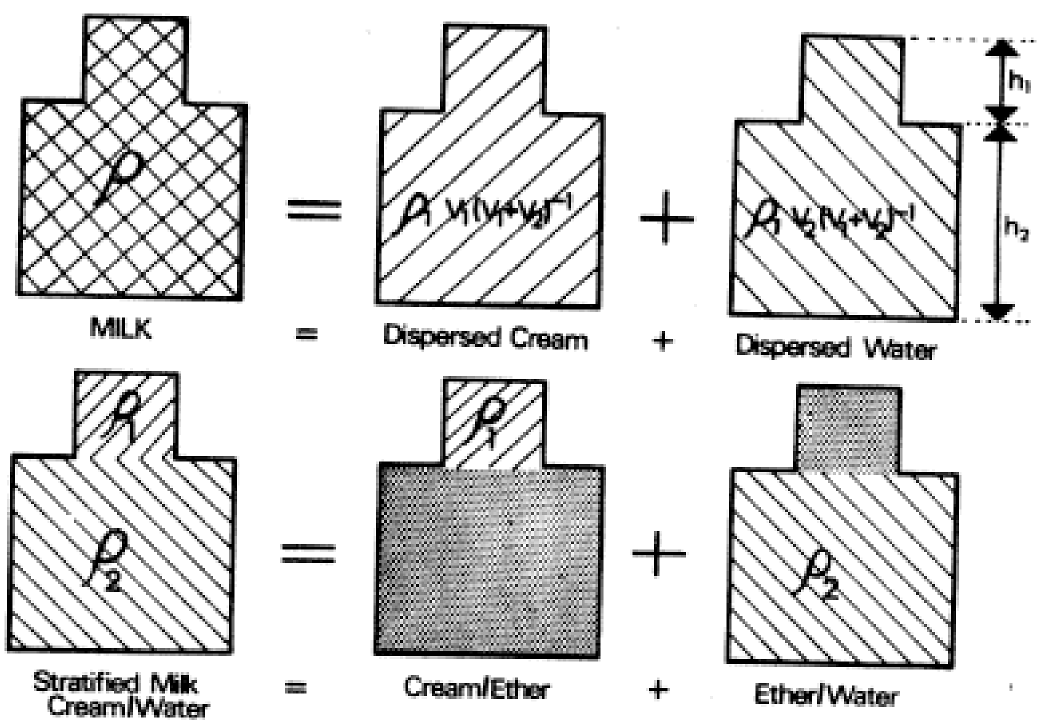
then $V_2 h_1 - V_1 h_2 = h_1 h_2 (A_2 - A_1) > 0.$

Thus as water is denser than cream, we have

$$p - p' = (p_{cream} - p'_{cream}) + (p_{water} - p'_{water})$$

$$= (\rho_2 - \rho_1)(V_1 + V_2)^{-1} (V_2 h_1 - V_1 h_2)$$

Fig (x) "Add Effects" decomposition applied to the "mathematician's milk bottle".



Thus, for a "mathematical milk bottle" we have established that the base pressure (p) drops to the value p' after separation of cream. Presentation of this more sophisticated derivation to students leaves for them the mere puzzle of extending the derivation to milk bottles of conventional shape. In fact the argument given above applies at once to a conventional bottle provided cream/milk volumes and vertical heights satisfy the inequality

$$V_2 h_1 - V_1 h_2 > 0$$

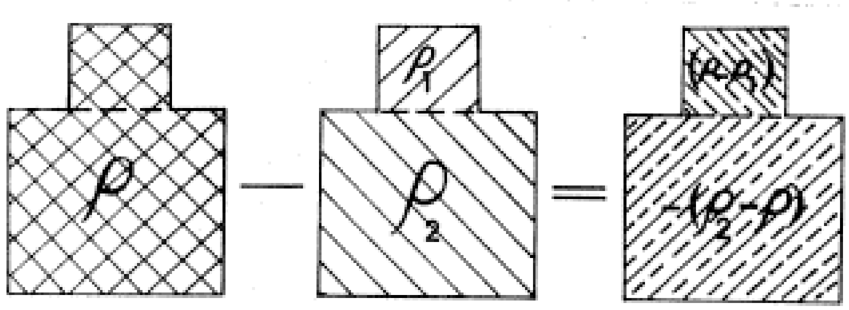
i.e.,

$$\frac{V_2}{h_2} > \frac{V_1}{h_1}$$

which is a requirement on the <u>average</u> cross-sectional areas.

"Subtract-Effects"

This heuristic is conceived by the writer as a variant of "Add Effects" discussed above. A "Subtract-Effects" motivated calculation of the differences in base pressures, p - p', is outlined in visual terms by Fig.(xi). Now in this figure we have not introduced a "pressure-ether" - - but the lower volume V_2 of the mathematical milk bottle now contains a liquid of negative density!



p p' p - p'

Fig (xi) Schematic outline of "Subtract Effects" motivated attack on MILKO.

Process

"Process" is a heuristic of great power which involves the notion of a state. From the "Process" viewpoint, a problem is conceived as devolving on a transformation, like so

$$(\text{State A}) \longrightarrow (\text{State B})$$

or, in short hand, A B. In terms of the parameters that define a state, the transformation is

$$A \rightarrow B \Leftrightarrow (a_1, a_2, a_3, \ldots) \rightarrow (b_1, b_2, b_3, \ldots)$$

The key problem solving idea of "Process" is to devise some (possibly fictitious) state X, for which the transformation rules

$$A \longrightarrow X \quad ; \quad X \longrightarrow B$$

are well established, so that one can readily compute the transformations of parameters,

$$(a_1, a_2, a_3, \ldots) \longrightarrow (x_1, x_2, x_3, \ldots) \longrightarrow (b_1, b_2, b_3, \ldots)$$

What has been presented above is a very sophisticated and formal description of "Process." In fact the present writer first identified this heuristic as being potent in thermodynamics and special relativity and conceived of this problem solving idea as being used and developed only by advanced students. However, in September 1974, I was flabbergasted to observe a five year old, Leo, use this very same heuristic. At the conclusion of a classic

Piagetian interview described in the Appendix, Leo was asked:

> "How would you explain to another child why the Pepsi (poured from a squat beaker) rises so high after pouring (into a narrow cylinder)?"

Leo thought intently for a few seconds, then answered,

> "The sides are pushing the Pepsi up".

Leo placed his hands apart and forward, then brought them together as he said this. It was clear in context that he had invented a fictitious "state X: in which the tall cylinder had the same diameter as the (squat) beaker, and therefore would hold its aliquot of Pepsi at the same level as that in the beaker. Leo's explanation entailed the transformation from

State A: Pepsi in squat beaker

to the final state

State B: Pepsi in tall narrow cylinder

via the fictitious State X.

Looking at the Dragon MILKO in "Process" terms, one perceives this Dragon as involving a transformation from

State A: Homogenous milk in milk bottle

to

State B: Stratified milk in milk bottle

One can't compute the alteration in base pressure -- i.e., $p_A - p_B = p - p'$ directly -- after all, this is <u>the</u> problem of this Dragon. Yet if the neck of the milk bottle were rubber. or were hinged somehow, and the bottle transformed into a cylinder it would be easy, in fact trivial, to compute the base pressure change after stratification by reference to the states:

State X: Homogenous milk in cylinder

State Y: Stratified milk in cylinder

In a cylinder the only vertical forces acting on the fluid contents (of weight W) are gravity and the base pressure acting over the area A, so that

$$p_X = p_Y = W/A$$

The additional base pressure in state a compared to state X is due to an additional height D of milk so that under the transformation A → X: $p_A - p_X = p - W/A = D\rho g$

Likewise: Y → B: $p_Y - p_B = W/A - p' = - D\rho_1 g$

Hence

$$p - p' = p_A - p_B = D(\rho - \rho_1)g$$

which is positive as cream density ρ_1 is less than the density ρ of milk. This "Process" argument is illustrated in Fig.(xii)

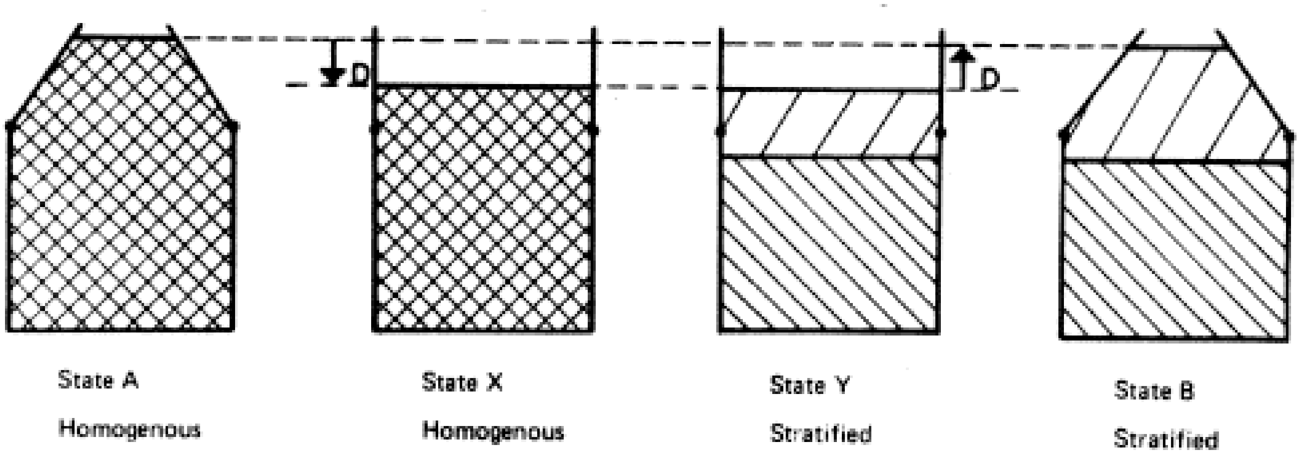


Fig (xii) "Process" applied to MILKO. The sides of the milk bottle are drawn as hinged.

It's worth noting an unsuccessful "Process" motivated attack on MILKO that a number of students initiate. Suppose the milk bottle is connect near its base with a vertical cylinder, as drawn in Fig.(xiii).The level of homogenous milk is equal in the two branches at the initial state. Subsequently the milk stratifies; however there are unequal lengths of strata in the two connected vessels, and there is no convenient intermediate state.

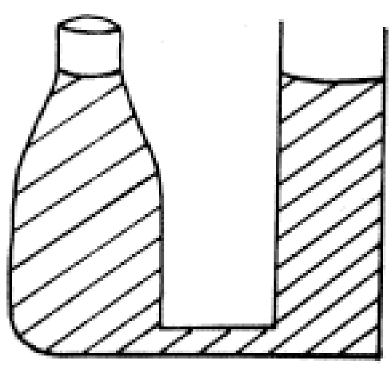


Fig (xiii) Sketch for an unsuccessful "Process" foray at MILKO.

JUGGLO

A certain juggler approached a flimsy bridge. He learnt that the bridge was barely strong enough to support his weight and the weight of just two of his juggling balls.
However, undeterred, the juggler set forth across, with three balls. By constant juggling there were always two balls in the air at any given instant.
Did he reach the other side?

Fig (xiv) The Dragon JUGGLO. Reproduced with permission from H.A. Cohen: "A Dragon Hunter's Box", Hanging Lake Books, Warrandyte, Victoria, 3113, Australia (1974).

2- JUGGLO

Preliminary

The dragon Jugglo of Fig (xiv) is a superlative Fermi problem that appears to have been first posed sometime in the last century. On being exposed to this Dragon a typical La Trobe undergraduate will answer firmly "No!" With some coaxing he will recount the thinking that lead to this conclusion. A typical response - - very much refined for these didactic purposes - - goes like so: "The bridge has a safe load of Mg + 2mg and presumably will collapse if this load is exceeded. It's supporting a Juggler and 3 balls of total weight Mg + 3mg. The balls are in the air sometime, and the juggling details are too horribly complicated even to envisage - - but the real point is that you've got a whole system (juggler plus balls) of a weight which exceeds the critical load - - so that the bridge collapses."

The basis for the correct "physical intuition" - - the response "No" as revealed by such verbalizing - - lies in the mechanical implications of the heuristic "In Toto."

However, if the tutor reformulates Jugglo, supposing that there are only two balls in all which the juggler is tossing on the very same bridge, incorrect solutions are common, if not so invariable as in the case of MILKO.

Now to get down to the slaughter of JUGGLO. Here are four different attacks - - named in accord with their dominant heuristic.

In Toto

We previously discussed the heuristic "In Toto" as applied to the Dragon MILKO. A familiar application of this problem solving schema is to the description of rigid bodies where the concept of the center of mass is introduced. To apply this idea to Jugglo involves considering the system of Juggler (of mass M) and N balls of mass m as a single object of mass M + Nm.

Students often adopt an "In Toto" viewpoint to examine JUGGLO - - a but in midstream seem to switch heuristic - - following the flag

(c) Switch to Divide and Conquer Viewpoint.

It seems that there is a particular debug routine attached to the "In Toto" frame

(d) Check the relation of the parts to the whole

that is easily confusable with having switched to "Divide and Conquer." In fact we use such a debug routine (d) to extend the fairly crude "In Toto" argument given above to the following polished attack on JUGGLO.

How do the component parts of the whole JUGGLO system interact? The answer is reassuring to the "In Toto" champion: the "internal" forces between the components are equal and opposite, and therefore of no consequence in considering the motion of the system in terms of the behaviour of the centre of mass. The various "external" forces, including the (upward) reaction of the bridge $\underline{R}$, have a sum of magnitude.

$$\underline{R} - (M + Nm)g\underline{k}$$

where $\underline{k}$ is a unit vertical (upwards pointing) vector. The center of mass of the system moves up and down a little, about some average position (or perhaps remains stationary). Consequently, if at any instant the center of mass is experiencing an upwards acceleration, then at that instant

$$R > (M + Nm)g.$$

Thus even in the case of two balls (N =2), if the center of mass of the system comprising Juggler and balls is not stationary, then at some instant there will be a net upward acceleration and the bridge load limit will be exceeded.

Divide and Conquer

A "Divide and Conquer" approach to a problem is to break the problem into interfacing problems, each of which is solved in turn. Applied to JUGGLO this heuristic would naturally lead us to consider separately the dynamics of the bridge, the juggler, and each of the three balls. Now the bridge is specified as capable of supporting a maximum load of (M + 2m)g - M being the mass of the juggler and m the mass of the ball. The first subproblem - - the juggler - - is easily analysed to deduce that the maxim force the juggler can exert on one (or more) balls at any instant is 2mg upwards. The next subproblem is the motion of one ball, ball I say. If at time t = 0 the ball is released with upward velocity v it will rise a distance (v/2g) in time v/g, and after a time lapse of 2v/g will return to the altitude of release, but now with downward velocity v. If caught at the same height as when released, then (presuming the juggler has no other balls in his hands at the time) the juggler can apply (maximum) upward force 2mg on the ball, so that the net force on the ball is mg upwards - - leading to a symmetric reversal of the motion as per Fig (xv).

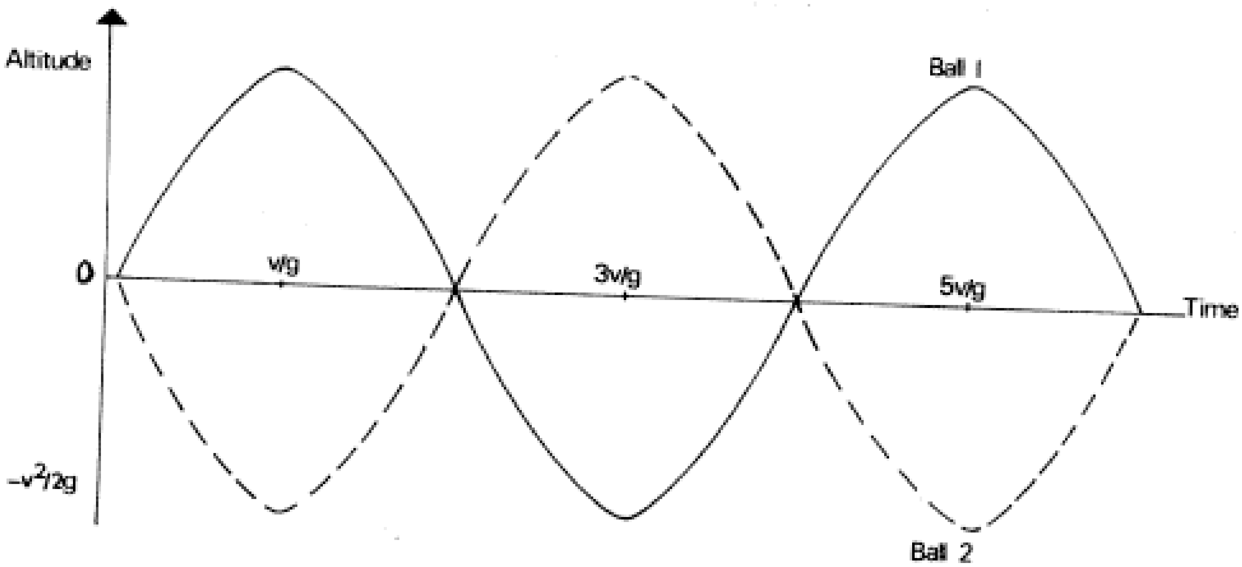


Fig (xv) Possible motions of two balls tossed by the juggler

In fig (xv) we've drawn not only the altitude of ball 1 assuming release and capture of this ball occur at a constant height, but also a permitted motion of ball 2. Clearly in accord with this analysis at all times the juggler is applying the maximum allowed force so that there is no possibility of him catching a further ball; there can be no Ball 3 without exceeding the bridge load limit.

The chief virtue of this "Divide and Conquer" attack h is the very detailed information derived as to an acceptable juggling style for two balls; if the greatest height reached by a ball was h ($= v^2/2g$) above catching level (marked 0 in Fig (xv)), the ball will fall a further h encased in the juggler's hand and then be brought up to be released at the catching level whilst simultaneously the second ball is caught - - possibly with the other hand at a different altitude - - after the second ball has likewise fallen through h.

Divide and Conquer ("Time Average" Algorithm)

This approach to JUGGLO is also motivated by "Divide and Conquer." However, the trick of taking a time average (such as is often done in statistical mechanics) is used to get rid of uninteresting dynamical detail.

Consider the equation of motion for ball a:

$$m\dot{\underline{v}}_a = -mg\underline{k} + \underline{F}_{aj}(t)$$

where as elswhere $\underline{k}$ is a unit vector in the vertical direction, and F_{aj} (t) is the force applied by the juggler to ball a at time t. Integreting between the limits t = 0to t =T,

$$m\underline{v}_a(T) - m\underline{v}_a(0) = -mgT\underline{k} + \int_0^T dt\, \underline{F}_{aj}(t)$$

Hence, the time-averaged value of the force $\underline{F}_{aj}$ is

$$\langle F_{aj} \rangle = \frac{1}{T}\int_0^T dt\, \underline{F}_{aj}(t) = mg\underline{k} + \frac{m\underline{v}_a(T) - m\underline{v}_a(0)}{T}$$

Provided this ball isn't dropped, the numerator of the second term on the left is bounded, so that over an extensive duration the time of $\underline{F}_{aj}$ is

$$\langle \underline{F}_{aj} \rangle = mg\underline{k}$$

Summing the forces on the juggler, and then considering the load on the bridge, gives for the time-averaged load on the bridge in the case of three balls,

$$\langle \underline{R} \rangle = Mg\underline{k} + 3mg\underline{k}$$

which exceeds the prescribed limit. For two balls

$$\langle \underline{R} \rangle = Mg\underline{k} + 2mg\underline{k}$$

Hence, if R is constant, the bridge is just safely loaded, but if R varies then at some instant it must exceed its average value.

"Add Effects"

In accord with the heuristic "Add Effects" we conceive the load on the bridge as being the cumulative (additive) effect of each of Jugglo, and 3 balls, teated separately. Thus the bridge is "held responsible" for on the average keeping each of these four objects above the bridge. The Juggler needs Mg to stay more or less where he is, and likewise each ball requires an external force of average mg to be on the whole uninfluenced by gravity. Hence the safe load is exceeded by a Juggler tossing three balls.

This informal discussion under this heading differs in small but crucial emphases from that given under the heading of "In Toto." The formal mathematical argument motivated by "Add Effects" is likewise similar to that given above under the heading "In Toto".

4- CONCLUSION

We have shown how a diversity of "solutions" to the Dragons MILKO and JUGGLO depend on just a limited number of problem solving schemata called heuristics. The core idea of these heuristics is probably acquired in childhood, but during intellectual development a coterie of debug routines, caveats, flags, problem transformation and reduction, ideas become attached to each heuristic. Knowledge of very specific skills termed algorithms is also linked with particular heuristics.

In order to promote student self-awareness, of the processes involved in their own intellectual development, and of the evolutionary character of the formulation of a solution to a formidable problem, a teaching stratagem is proposed with the following facets:

1) Specific discussion with students of the model for problem solving and for intellectual development presented here.
2) The posing to students of really formidable challenging problems, of which the two Dragons discussed herein are instances.
3) The discussion of student forays at Dragons with students in order that their own attempts can be interpreted in terms of the theoretical framework provided by the concept of the elaboration of heuristics.

ACKNOWLEDGEMENTS

This paper had its origins in the recording of a large number of interviews in which the interviewees tried to slay Dragons despite my suggestions and countersuggestions: to those students and colleagues who participated I am indeed grateful. The Division for Study and Research in Education at M.I.T. provided a stimulating environment in which to clarify my thoughts and write the first draft : I would thank Prof. W.T. Martin for the invitation and support of the Division. It is a pleasure to recall the special stimulus of discussions with Seymour Papert and Marvin Minsky at Cambridge and with Henry Krips at Melbourne.

Appendix

This paper as a whole has been concerned with the development of problem solving ability in physics. However the teaching stratagem I espouse is based on a theoretical model of intellectual development that has far wider gambit. In this Appendix the model is applied to give an explication of certain aspects of the intellectual development of children, by showing how it interprets some of the data obtained in the "protocols" (transcripts) of three Piagetian experiments.

The Egg and Egg Cup Experiment

In order to answer questions as to whether some quantity is greater or less than another, the typical child uses such heuristics as

H1. Greater extent means more.

H2. Sparseness (greater gaps between elements) means less.

H3. Counting you if more or less.

The heuristic H3 is only suitable for very small sets because of a child's limited skill at counting. The sorts of situations where a typical child of five years gives the correct answer to questions about quantity are shown in fig(xvi), A being what we term a paradigm for H1, while B is a paradigm for H2. It is noteable re these two paradigms that only one heuristic is applicable to each paradigm: but what happens if a situation is presented to which both heuristics are applicable, and give conflicting conclusions? In one of the classic "conservation" experiments of Jean Piaget[12], the egg-cup experiment, children in the 4 - 7 years

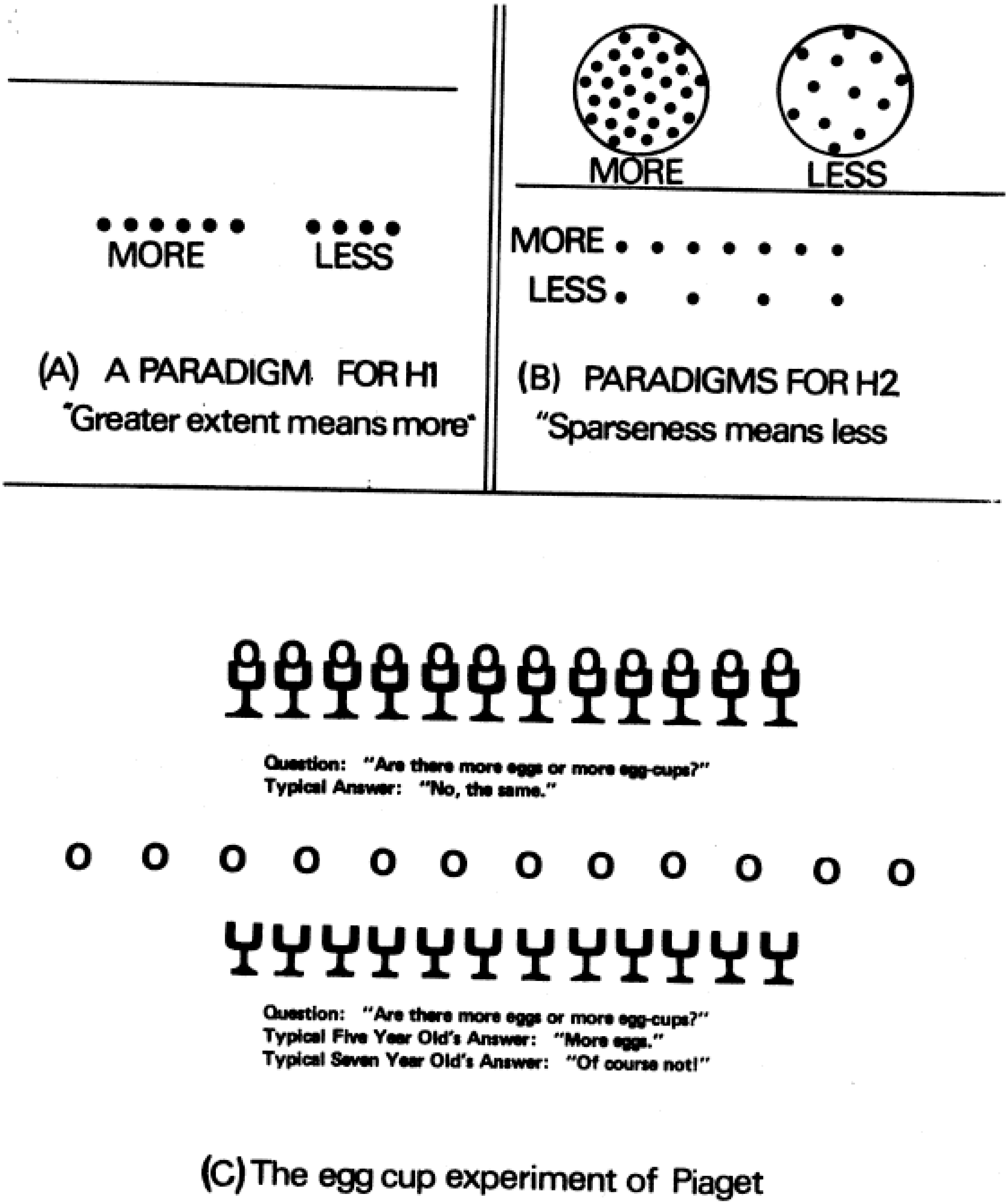

MORE
LESS
(A) A PARADIGM FOR H1
"Greater extent means more"
MORE
LESS
MORE
LESS
(B) PARADIGMS FOR H2
"Sparseness means less
Question: "Are there more eggs or more egg-cups?"
Typical Answer: "No, the same."
Question: "Are there more eggs or more egg-cups?"
Typical Five Year Old's Answer: "More eggs."
Typical Seven Year Old's Answer: "Of course not!"
(C) The egg cup experiment of Piaget

Fig (XVI)

age group are set such a puzzle involving a clash in their heuristics. As indicated in fig (xviC) if such children are shown a line of eggs in egg-cups, where the extent and sparseness of both the eggs and the egg-cups are the same, then in answer to the question "Are there more eggs or more egg cups?" the typical child (4 - 7 years) answers "No the same." However, if these eggs are in full view of the child removed from the cups - - and spread out in a longer line than the line of the cups - - then the situation is one in which H1 and H2 give conflicting assessments to the repeated question. However, for the young child, H1 is in some way tagged as primary or more important - - for, as indicated below, H1 describes a great range of situations where such evaluations are sought. So the typical five year old will now reply "Of course not." What distinguishes the seven year old from the typical five year old? Possibly the seven year old has acquired a heuristic such as

H4. Relationships more than or less constant in time.

which certainly doesn't adequately describe the contents of a cookie jar but nevertheless is a valuable heuristic. However the mere addition of H4 to a child's reportoire won't necessarily lead to the correct answer to the repeated question of the egg egg-cup experiment. What is needed is some caveat like

H5. In case of conflict between H1 and H2, use an historical heuristic like H4.

The addition of these - - or some such - - heuristics to the collection

H1, H2 etc. of the typical five year old child is an instance of what I term the debugging of heuristics.

Another Piagetian "Conservation" Experiment

Here is the protocol of a classic Piagetian "conservation" experiment, conducted by one of Piaget and Inhelder's collaborators, Olivier de Marcellus, in Lexington, Massachusetts in September, 1974.

A five year old child Rob was shown two vessels. One, a measuring cylinder, was tall and narrow in cross-section, the other was a squat beaker containing a dark liquid termed "Pepsi". Rob was asked to what height he anticipated the "Pepsi" poured from the beaker would fill the narrow cylinder.

Rob pointed to a level on the cylinder at the same height *(1) as the top level of the "Pepsi" in the squat vessel. The "Pepsi" was poured. The level in the narrow cylinder was about three times higher than that predicted by Rob. Rob registered much astonishment, followed by traditional facial expressions for grasping a tricky idea.
Rob was asked: "Is there more Pepsi now?"
Rob replied: "No! It's just the same ... it only looks more."*(2)
Rob was then asked how he would explain to another child how it was that the "Pepsi" was so high in the (narrow) cylinder.
Rob pondered a moment -- then placed his hands about 20 centimetres apart in front of him, then steadily drew his hands together while saying, "The sides are pushing the Pepsi up".*(3)
Rob's responses, *(1), *(2), and *(3) of the above protocol, merit these comments:

*(1) Rob s expectation of the height of the new (narrow) liquid column conforms to the heuristic H1 of the preceding experiment -- the anticipated "extent" of the new "Pepsi" column -- its height -- was anticipated to be unchanged.

*(2) Rob opined a caveat to be referred to as H6 which he probably only recently learnt to associate with the heuristic H1.

H6 : "Sometimes it only looks more".

On the basis of this protocol one can't be certain as to which heuristic(s) led Rob to say "the same" -- but it was probably the historical heuristic H4 delimited above.

*(3) Rob had formulated an explanation in terms of the heuristic <u>Process</u> -- the same heuristic, which somewhat elaborated (= debugged) was used to snare the Dragon Milko in Section 2. Rob was considering a fictitious state of the cylinder -- presumably one in which cross-section was the same as in the squat beaker. In the fictitious state, the "Pepsi" would be at the same level as in the squat beaker. But on bringing the sides together -- as indicated by hand movement -- the "Pepsi" level would rise.

Islands Experiment

The following incident took place within the context of a very extensive Piagetian experiment,"Islands",conducted by

Seymour Papert.

A five year old child was asked to count the (2 cm. x 2 cm. x 2 cm.) cubes arranged as a rectangular prism which was 8 cm. x 6 cm. x 6 cm.. Her algorithm was transparent, as she traced her finger row by row along the front face, and proceeded to likewise count blocks on other faces of the prism. she concluded there were 30 cubes in the prism. She was asked "How did you do it? If another child wanted to count the blocks, what would you tell her?" The child replied, "Don't count the side (= edge) one's twice." The child failed to say that her basic method was systematically tracing her finger along the faces. This method had the bug she discovered (as well as others she didn't discover - - the inner blocks weren't counted) but the counting procedure is not well characterised heuristically by that bug!

In applying the heuristic "Add Effects" to Milko in Section 2 a similar situation arises. An incredible artifact, a pressure carry ether, has to be introduced for this heuristic to succeed. Yet it would be patently misleading to heuristically characterise this solution as "The Pressure-Ether Model" for the Milko Dragon.

Interpretation of "Conservation" in terms of Heuristic Frames

In describing above some classic Piagetian "Conservation" experiments[16] we have noted the heuristics manifestly utilised - and in some instances verbally expressed by children in the five to seven years age group. Perhaps we should note that it is fairly novel to attempt to use the protocols of such experiments to determine the heuristics reportoire of a child: such a discussion was first given by Seymour Papert.[17] The evidence of these and other protocols suggest that a child does not mature by discarding the "non-conserving" heuristics and learning a more "precise" "conserving" heuristic: rather to the prototype heuristic "To tell if more - look" are added further structural elements -- other heuristics - the whole collection of heuristics being closely linked, and heuristics relating the various elements are part of the whole. Table III shows how some of the heuristics discussed above slot into the Heuristic Frame which is called "Look - More".

TABLE III

THE ANATOMY OF THE HEURISTIC FRAME "LOOK-MORE"

COMPONENT	SPECIFICATION
Core heuristic	"To tell if more - look"
Problem Reduction Devices and Algorithm Selector	H1: "Greater extent means more" H2: "Sparseness means less"
Debug routines	"Check H1 and H2 for consistency"
Demons	H5: "In case of conflict between H1 and H2, use an historical heuristic" H6: "Sometimes it only looks more"

The <u>Heuristic Frame</u> detailed in Table II is similar to a schema proposed by Minsky and Papert.[19] The young child has available the core heuristic of this frame - the idea that visual observation can be used to determine quantity - plus H1 and possibly H2. Of course to a young child quantity means capacity to satisfy hunger or maybe bites. One of the most endearing protocols I have collected was of a non-conserving six year old, who was asked whether a flattened ball of dough contained more than a spherical ball which had previously been adjudged "the same amount". The girl guided by H1 claimed that the flattened ball of dough contained more, and justified this answer by pointing out that the round ball could be eaten in two bites, whereas the flattened ball would take five bites. The older child - the Conserver - has added to these basic elements of a frame debug routines and demons akin to those in the Table. It is just that process of augmenting and editing a frame, such as "Look - More", which is called in this paper the <u>debugging</u> of that <u>heuristic</u>.

REFERENCES

1. Jearl Walker, "The Flying Circus of Physics", John Wiley and Sons, New York, 1975.

2. H.A. Cohen, "A Dragon Hunter's Box", Hanging Lake Books, Warrandyte, Victoria, 3083, Australia (1974).

3. G. Polya, "How to Solve It", 2nd edition, Double Bay, Garden City, N.Y. (1957). "Mathematics and Plausible Reasoning". Princeton University Press, Princeton, N.J. (1954).

4. The sort of solution that Fermi anticipated for this dragon would be something like so: There are 8 million people in New York. 80% say of pianos are in family homes and apartments, of which one can estimate ... if there is one such unit per 5 people as 1.6 million. Such and such a fraction of homes possess of piano. A piano needs tuning after such a period ... Piano owners perceive a piano needs tuning after a further time-lapse, or when certain pitches are significantly in error. To tune a piano takes an estimated amount of time, so that to tune the requisite weekly number of pianos require so many piano tuners working 35 hours per week, including travelling and administrative time ...

5. H.A. Cohen, "What G Killed Ned Kelly? and Other Problematical Dragons". Published by the author, Melbourne, Australia, (1972).

6. To claim that Dragons or any other problems have the flavour of research requires for justification a detailed discussion of the pattern of scientific progress. The

philosophers of science Hanson, Kuhn and Lakatos are especially relevant in this regard. See Refs 7) 8) 9) 10) 11).

7. I. Lakatos, "Proofs and Refutations", British Journal of Philosophy and Science, 14, 1-25, 120-39, 221-43, 296-342 (1963/64).

8. N.R. Hanson, "Patterns of Discovery", Cambridge Univ. Press, England (1958).
N.R. Hanson, "Perception and Discovery", Freeman, Cooper and Co., San Francisco (1969), "Observation and Explanation", Harper Row and Co., New York, (1971).
T.S. Kuhn, "The structure of Scientific Revolutions", University of Chicago Press (1962).

9. I. Lakatos, "Falsification and the Methodology of Scientific Research Programs" in I. Lakatos and A. Musgrave (Editors) Criticism and the Growth of Knowledge, Cambridge University Press, England (1970). Also see I. Lakatos in R.C.Buck and R.C. Cohen (Editors), Boston Studies in the Philosophy of Science Vol. 8 (1972). Dordrecht Reidel.

10. H.A. Cohen, "Mathematical Dragon Hunting on the La Trobe Campus". Australian Vice-Chancellor's Committee Educational Newsletter, No 3/73, Published by the A.V.C.C., Canberra, Australia (1973).

11. H.A. Cohen, "The True Confessions of an Aunt Watcher" scheduled for publication in the A.V.C.C. Educational Newsletter (cf. Ref. 10).

12. The study of human problem solving is the domain of cognitive psychology. My own work relates especially to the pioneering work of Piaget (ref 14) in being concerned with the evolution of intellectual capacities ("genetic

epistemology"). The frames concept presented here is a way of organizing data structures describing procedural knowledge possessed by humans , but would also provide the qualitative outline of a fully computable model of the problem solving process. The overall project is comparable to the efforts of Newell, Simon and co-workers at Carnegie-Mellon University who have collected many protocols and developed computer models of adult efforts to solve problems in crypto-arithmetic , chess end-games, logic puzzles and the like. See A. Newell and H.A. Simon, "Human Problem Solving", Prentice Hall, Englewood Cliffs, N.J. (1972).

13. The term "Frame" has been borrowed from Marvin Minsky "Whenever one encounters a new situation (or makes a substantial change in one's viewpoint) he selects from memory a structure called a <u>frame</u>; a remembered framework to be adapted to fit reality by changing details as necessary."

M. Minsky, "A Framework for Representing Knowledge" M.I.T. Artificial Intelligence Laboratory Memo No.306 (1974). To be republished in "The Psychology of Computer Vision" (McGraw Hill (1975).

14. J. Piaget, "The Child's Conception of Number" Norton, New York (1965) J. Piaget and B. Inhelder, "The Child's Conception of Space", Norton, New York (1967).

15. Professor Feynman has confirmed the substance of this account in a private communication.

16. See p.148, Galileo Galilei, "Dialogues Concerning Two New Sciences". Translated by H. Crew and A. de Salvio, Dover, N.Y. (1954).

17. This point is discussed from a philosophical perspective in Chapter 7 entitled "Extensive Magnitudes" in Rudolph Carnap, "Philosophical Foundations of Physics" (Edited by Martin Gardner), Basic Books (1966).

18. Seymour Papert, "The Language of children and the language of computers". Estratto da Linguaggi nella societa e nella tecnica (1970), p.417.

19. Marvin Minsky and Seymour Papert, "Progress Report" Artificial Intelligence Memo No. 252 (1972). See Section 4.1.

20. It is worth the space of a footnote to recount one effort by a psychologist (who was a collaborator of Piaget) to "teach children the correct answer" to Piagetian puzzles like the above.[18] The teaching consisted in informing the child of the "right" answer to the question asked in a certain puzzle. The effect found was that a child at a certain level of development might be so taught to say "more" or "less" correctly in certain of what Piaget terms "conservation" experiments ... but not over an extensive range of experiments. Moreover, one can ask of a child who has said of some quantity that it is more than some other quantity to image the substance being considered was really "yummy", and then to say which lot he would prefer to eat: his heuristic judgement is then manifest despite his use of the appropriate <u>verbal formula</u>.. In fact the child who had been linguistically trained to give the right answer had merely been exercised in formula cranking, and though he might give the appropriate verbal response, his preference for a particular quantity of "yummy" substance would establish

that his qualitative judgement had not been altered by his subjection to a "teaching" program that was not concerned with the debugging of the child's <u>heuristic</u> frames. Now it is my assertion that drilling students only in standardised problems -- which I've termed formula crankers -- can, despite the best of intentions, be a very similar teaching program: our students can then "do" correctly just a limited range of problems, but don't possess property debugged <u>heuristics</u> capable of snaring a Dragon.

See p.48, J. Piaget, "Genetic Epistemology", Norton, New York (1970).

21. In his recent note, entitled "Painless precession", Eastman gave just such a "Divide and Conquer" motivated approach to precessional problems. This note also lists previous A.J.P. discussions of this 'mystifying' phenomena. See P.C. Eastman, Am. J. Phys. 43, 366 (1975).