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THE EMPTINESS AND COMPLEMENTATION PROBLEMS FOR AUTOMATA ON INFINITE TREES

BY

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ABSTRACT

In [6] Rabin defines Automata on Infinite Trees, and the body of that paper is concerned with proving two theorems about these automata.

The result we consider in the first chapter says that there exists an effective procedure to determine, given an automaton on infinite trees, whether or not it accepts anything at all. We present a new decision procedure which is much simpler than Rabin's since we do not use an induction argument as he does. We show in Theorem 1, the main theorem of Chapter 1, that if $\mathcal{A}$ is an automaton on infinite trees, then $T(\mathcal{A})$ (the set accepted by $\mathcal{A}$) is non-empty if and only if there exists a finite tree E and a run of $\mathcal{A}$ on E of a particular type. This latter condition is equivalent to saying that the set accepted by a particular automaton on finite trees is non-empty. Hence (see Theorem 2) the emptiness problem for automata on infinite trees can be reduced by Theorem 1 to the emptiness problem for automata on finite trees, which is shown decidable in [7]. Theorem 1 is proven by showing how maps on finite trees can generate maps on infinite trees which are then said to be finitely-generable. A corollary of the proof of Theorem 1 is that if an automaton on infinite trees accepts some input tree, then it accepts a finitely-generable one; this result was proved in a much more complicated way by Rabin in [5].

Chapter 2 is concerned with the more difficult result of [6] that for every automaton on infinite trees, $\mathcal{A}$, there exists another one, $\mathcal{A}'$, such that $\mathcal{A}'$ accepts precisely the complement of the set accepted by $\mathcal{A}$. Rabin's construction of $\mathcal{A}'$ and the proof that it works is an involved induction. In this paper we present a fairly simple description of a complement machine $\mathcal{A}'$, given $\mathcal{A}$, such that it is very plausible that $\mathcal{A}'$ works in the sense that $T(\mathcal{A}')=\overline{T(\mathcal{A})}$. The proof that our construction works, however, is difficult and very similar in complexity to Rabin's proof in [6] that his (more difficult) construction works.

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TABLE OF CONTENTS

CHAPTER 1
The Emptiness Problem

Section 1: Introduction

The analysis of finite automata on infinite trees is the basis for Rabin's remarkable proof of the decidability of S2S (the monadic second-order theory of two successors) [6]. Rabin's proof follows the now standard form of Büchi and Elgot's proof for WS1S (weak, single successor) [1,3] and Thatcher-Wright's proof for weak S2S [7], and requires demonstrating effectively that the automata are closed under union, projection, and negation, and that the emptiness problem for the automata is decidable. As in the case of S1S, the main difficulty in the case of S2S lies in proving closure under complementation of sets accepted by nondeterministic automata on infinite trees. The problem is complicated by the fact that nondeterministic infinite tree automata are known not to be equivalent to any of the likely definitions of deterministic infinite tree automata.

In [6] Rabin shows how, given an automaton on infinite trees, $\mathcal{A}$, one can construct another one, $\mathcal{A}'$, such that $\mathcal{A}'$ accepts exactly the complement of the set accepted by $\mathcal{A}$. His construction, however, is a very complicated induction. In Chapter 2, we present a fairly simple construction for $\mathcal{A}'$.

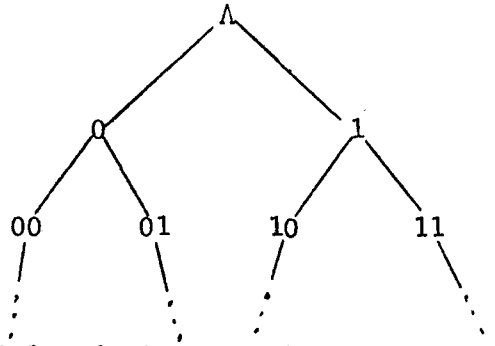
Curiously, the emptiness problem, which is easy for the other kinds of automata, turns out to be nontrivial for (nondeterministic) infinite tree automata. Rabin subsequently improved his original proof of the decidability of this emptiness problem, but even the second proof [5] uses an involved induction and consequently does not yield a simple

effective criterion for deciding emptiness.

In this chapter we provide such a criterion by showing that an infinite tree automaton accepts some valued tree if and only if there is a computation of the automaton containing a certain simple kind of finite subtree. Moreover, the set of finite subtrees of the kind we require is recognizable by a finite tree automaton, and in this way we reduce the emptiness problem for infinite tree automata directly to that for finite tree automata. This also yields a simple proof of another result of Rabin's about "regular" runs by automata (see below).

Section 2:

For this paper the appropriate way to visualize the infinite binary tree T is as follows. At the top is the root Λ. Every $x \in T$ has a left son $x \cdot 0$ and a right son $x \cdot 1$. Hence $T=\{0,1\}^*$.



We define a partial ordering on T by $x \leq y$ if $y=x \cdot z$ for some $z \in \{0,1\}^*$. If $x \leq y$ and $x \neq y$, then we will write $x < y$.

For each $x \in T$, define the (sub)tree with root x to be the set $T_x=\{y \mid x \leq y\}$. Thus $T=T_\Lambda$.

Definition: A path π of T_x is a set $\pi \subset T_x$ satisfying: 1) $x \in \pi$; 2) if $y \in \pi$, either $y \cdot 0 \in \pi$ or $y \cdot 1 \in \pi$ but not both; 3) π is a minimal subset of T_x satisfying 1) and 2).

Notation: If x and y are members of T and $x \leq y$, then we denote by $[x,y]$ the set $\{z \mid x \leq z \leq y\}$.

For a set B we denote the cardinality of B by $c(B)$ and the set of subsets of B by $P(B)$.

Definition: A set $B \subset T_x$ is called a frontier of T_x if for every path $\pi \subset T_x$ we have $c(\pi \cap B)=1$. By König's Lemma, every frontier of T_x is finite.

For $x \in T$, a finite tree with root x is a set $E_x=\{z \mid x \leq z \leq y$, for some $y \in B\}$, where B is a fixed frontier of T_x. For E_x as above, B is called the frontier of E_x and is denoted by $Ft(E_x)$. Unless otherwise noted, we will use E to denote a finite tree with root Λ.

Definition: A Σ-tree is a pair $t_x=(v,T_x)$ such that $v:T_x \to \Sigma$. A finite Σ-tree is a pair $e_x=(v,E_x)$ where $v:E_x \to \Sigma$. Unless otherwise noted, we will use t and e to denote t_Λ and e_Λ respectively. If $t=(v,T)$ is a Σ-tree, then we use both (v,T_x) and t_x to denote $(v \mid T_x, T_x)$. If $t=(v,T)$, and E_x is a finite tree, then we use e_x and (v,E_x) to denote $(v \mid E_x, E_x)$.

Definition: For a mapping $\theta:A \to B$, $In(\theta)=\{b \in B \mid c(\theta^{-1}(b)) \geq \omega\}$.

Definition: Let $\theta:A \to B$ and let $\Omega=(<L_i,U_i>)_{1 \leq i \leq n}$ be a finite sequence of pairs of finite sets. We say θ is of type Ω, written $\theta \in [\Omega]$, if for some i, $1 \leq i \leq n$, we have $In(\theta) \cap U_i \neq \phi$ and $In(\theta) \cap L_i = \phi$. If Ω is the empty sequence, then we define it never to be the case that $\theta \in [\Omega]$.

Definition: An f.a.t. (finite automaton on trees) is a system $\mathcal{A}=<S, \Sigma, M, s_0, \Omega>$ where S is a finite set of states, Σ is a finite set, $M:S \times \Sigma \to P(S \times S)$, $s_0 \in S$ is the initial state, and $\Omega=(<L_i,U_i>)_{1 \leq i \leq n}$ is a

finite sequence of pairs of finite sets.

If $t_x=(v,T_x)$ is a Σ-tree, then an $\mathcal{A}$-run on t_x is any mapping $r:T_x \to S$ such that: 1) $r(x)=s_0$ and
2) for all $y \in T_x$, $\langle r(y\cdot 0), r(y\cdot 1)\rangle \in M(r(y),v(y))$.

If $e_x=(v,E_x)$ is a finite Σ-tree, then an $\mathcal{A}$-run on e_x is any mapping $r:E_x \to S$ such that: 1) $r(x)=s_0$ and
2) for all $y \in E_x - Ft(E_x)$,
$\langle r(y\cdot 0), r(y\cdot 1)\rangle \in M(r(y),v(y))$.

The set of all $\mathcal{A}$-runs on t_x (e_x) will be denoted by $Rn(\mathcal{A},t_x)$ ($Rn(\mathcal{A},e_x)$, respectively). An accepting $\mathcal{A}$-run on t_x is any $r \in Rn(\mathcal{A},t_x)$ such that for every path $\pi \subset T_x$, $r|\pi \in [\Omega]$. Define $T(\mathcal{A})=\{t_x \mid$ there exists an accepting $\mathcal{A}$-run on $t_x\}$. $T(\mathcal{A})$ is called the set accepted by $\mathcal{A}$.

Given an f.a.t. $\mathcal{A}=\langle S, \Sigma, M, s_0, \Omega\rangle$, we wish to determine whether or not $T(\mathcal{A})=\phi$. Consider the automaton $\bar{\mathcal{A}}=\langle S,\{a\},\bar{M},s_0,\Omega\rangle$ where for all $s \in S$, $\bar{M}(s,a)=\bigcup_{\sigma \in \Sigma} M(s,\sigma)$. Clearly $T(\mathcal{A})=\phi \Leftrightarrow T(\bar{\mathcal{A}})=\phi$.

Thus, the emptiness problem is reduced to the case of automata over the single letter alphabet $\{a\}$. Henceforth in this section we restrict our attention to this case. Since there exists just one $\{a\}$-tree rooted at Λ, $(\bar{v},T)$, and for every finite tree E just one finite $\{a\}$-tree, $(\bar{v},E)$, we will omit mention of the valuation $\bar{v}$ and talk about $\mathcal{A}$-runs on T and E, $\mathcal{A}$ accepting T, etc. Clearly, $T(\mathcal{A})\neq\phi \Leftrightarrow T \in T(\mathcal{A})$.

Theorem 1: Let $\mathcal{A} = \langle S, \{a\}, M, s_0, ((L_i, U_i))_{1 \le i \le n} \rangle$ be an f.a.t. $T(\mathcal{A}) \neq \emptyset \Leftrightarrow$ for some finite tree E there exists an r such that

1) $r \in Rn(\mathcal{A}, E)$,

2) there exist mappings J: Ft(E) $\to$ E-Ft(E) and H: Ft(E) $\to$ E-Ft(E) such that for all $x \in Ft(E)$

a) $H(x) \le J(x) < x$,

b) $r(J(x)) = r(x)$,

c) $r([H(x), J(x)]) = r([J(x), x])$,

d) for some i, $1 \le i \le n$, $r([J(x), x]) \cap L_i = \emptyset$ and $r(x) \in U_i$.

Before we prove Theorem 1, we show that Theorem 1 easily yields the following theorem.

Theorem 2: The emptiness problem for f.a.t.'s is decidable.

Proof of Theorem 2: Let $\mathcal{A}$ be as in the statement of Theorem 1.

Definition: Let E be a tree (finite or infinite). Let r be an $\mathcal{A}$-run on E. Let $x \in E$. Since $x \in \{0,1\}^*$ we can write $x = \sigma_1\sigma_2 \ldots \sigma_m$. Define $\alpha_{r,x}$ to be the following member of S^*: $\alpha_{r,x} = r(\Lambda) \cdot r(\sigma_1) \cdot r(\sigma_1\sigma_2) \ldots r(x)$.

Notation: Let α be a finite string and let n and m be positive integers, $n \le m$. Then by $\alpha(n)$ we will mean the nth element (from the left) of α. By $\alpha([n,m])$ we will mean the set of elements between and including the nth and the mth places of α. Note that $\alpha(n)$ is only defined if $1 \le n \le \text{length}(\alpha)$ and $\alpha([n,m])$ is only defined if $1 \le n \le m \le \text{length}(\alpha)$.

Definition: Let $\alpha \in S^*$. We say that α is Good if there exist positive integers H and J such that $H \leq J < N = \text{length}(\alpha)$, $\alpha(J) = \alpha(N)$, $\alpha([H,J]) = \alpha([J,N])$, and there exists an i such that $\alpha(N) \in U_i$ and $\alpha([J,N]) \cap U_i = \emptyset$. Note that good is defined with respect to our f.a.t. $\mathcal{A}$.

Lemma 1: The set of good strings is a regular set, i.e., it is recognizable by a finite state machine on finite input strings.

Proof of Lemma 1: Obvious. □

Lemma 2: Let G be a regular set of finite strings on S. Let H = {E|E is a finite tree and there exists a run r on E such that for all $x \in Ft(E)$, $\alpha_{r,x} \in G$}. Then H is recognizable by a finite automaton on finite trees as defined in [7].

Proof of Lemma 2: Fairly obvious. □

Completion of proof of Theorem 2: By Theorem 1, Lemma 1, and Lemma 2, the emptiness problem for $\mathcal{A}$ can be reduced to the emptiness problem for a particular finite automaton on finite trees. But by Theorem 7 in [7], this problem is decidable.

Proof of $\Rightarrow$ in Theorem 1: Let r be an accepting $\mathcal{A}$-run on T. By the definition of accepting run and of good string, it is clear that for every path π of T there exists an x, $x \in \pi$, such that $\alpha_{r,x}$ is a good string. Let B = {x|$\alpha_{r,x}$ is good and for all $y < x$, $\alpha_{r,y}$ isn't good}. Then B is a frontier. If we let E be the finite tree with frontier B, then there exist mappings J and H which, together with r|E, satisfy conditions 1 and 2 of Theorem 1. This completes the proof of $\Rightarrow$.

Proof of $\Leftarrow$ in Theorem 1: Let E, r, J and H be as specified in 1) and 2) of Theorem 1.

We define a mapping η: $T \to E$ inductively as follows. Let $\eta(\Lambda) = \Lambda$. If $\eta(x)$ has been defined, then for $\sigma \in \{0,1\}$, define $\eta(x^\frown\sigma)$ as follows.

Case 1: If $\eta(x) \in E - Ft(E)$, then let $\eta(x^\frown\sigma) = \eta(x)^\frown\sigma$.

Case 2: If $\eta(x) \in Ft(E)$, then let $\eta(x^\frown\sigma) = J(\eta(x))^\frown\sigma$.

Define $\bar{r}$: $T \to S$ by $\bar{r}(x) = r(\eta(x))$, for all $x \in T$. Clearly by 2) b) of Theorem 1, $\bar{r} \in Rn(\sigma\!\!\!/, T)$ so that it suffices to show that for all paths $\pi \subset T$, $(\bar{r} \mid \pi) \in [\Omega]$, because then $T \in \mathbb{T}(\sigma\!\!\!/)$ and hence $T(\sigma\!\!\!/) \neq \phi$.

Let $\pi \subset T$ be a specific path. Let $y_0, y_1, y_2, \ldots$ be the infinite subset of π (listed in increasing order under $\leq$) consisting of exactly those members of π whose images under η are in Ft(E). Define V_π to be the following infinite sequence of members of Ft(E) $\times$ Ft(E):

$$V_\pi = \langle\eta(y_0), \eta(y_1)\rangle, \langle\eta(y_1), \eta(y_2)\rangle, \langle\eta(y_2), \eta(y_3)\rangle, \ldots$$

For all $i < \omega$ we have by the definition of η, $J(\eta(y_i)) \leq \eta(y_{i+1})$ and $\bar{r}([y_i, y_{i+1}]) = r([J(\eta(y_i)), \eta(y_{i+1})])$. Hence, $In(\bar{r} \mid \pi) =$

$$\bigcup_{\langle x,z\rangle \in In(V_\pi)} r([J(x), z]).$$

Clearly there exists a finite sequence (possibly with repetition) of members of Ft(E), $x_1, x_2, x_3, \ldots, x_m$, such that

$$x_1 = x_m \text{ and}$$

$$\text{(I)} \qquad In(V_\pi) = \{\langle x_1, x_2\rangle, \langle x_2, x_3\rangle, \langle x_3, x_4\rangle, \ldots, \langle x_{m-1}, x_m\rangle\}.$$

From now on we will denote $J(x_i)$ by J_i and $H(x_i)$ by H_i, for all $1 \le i \le m$.

We have from the preceding paragraph

$$\text{(II)} \qquad \begin{array}{l} \text{for all } 1 \le i < m,\ J_i < x_{i+1}, \\ \text{and } In(\bar{r} \mid \pi) = \bigcup_{i=1}^{m-1} r([J_i, x_{i+1}]) \end{array}$$

$(\bar{r} \mid \pi) \in [\Omega]$ is immediate from the third of the following three lemmas.

Lemma 3: There exists an M, $1 \le M \le m$, such that for all i, $1 \le i \le m$,

$$H_M \le H_i.$$

That is, $H_M = \min\{H_1, \ldots, H_m\}$.

Proof: Our induction hypothesis at stage h is that there exists an integer M', $1 \le M' \le h$, such that for all i, $1 \le i \le h$, $H_{M'} \le H_i$. Clearly the basis case is trivial. We assume the induction hypothesis for h and prove it for h+1.

$H_{M'} \leq H_h$, by the induction hypothesis.

$H_h \leq J_h$, by 2) a) in Theorem 1.

$J_h < x_{h+1}$, by (II).

Hence, $H_{M'} < x_{h+1}$.

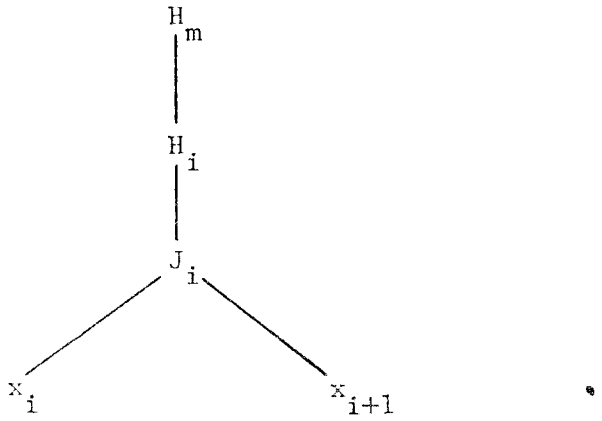
By 2) a) of Theorem 1 we also have $H_{h+1} < x_{h+1}$. Therefore, H_M and H_{h+1} are comparable (under $\leq$). Clearly for all i, $1 \leq i \leq h+1$, $\underline{\min}\{H_{M'}, H_{h+1}\} \leq H_i$. □

If $M \neq m$, we can rename $x_1, x_2, \ldots, x_m$ so that (I) and (II) remain true and $H_m = \underline{\min}\{H_1, \ldots, H_m\}$. Henceforth, without loss of generality we assume that $M = m$.

Lemma 4: If $H_m = \underline{\min}\{H_1, \ldots, H_m\}$, then for all i, $1 \leq i \leq (m-1)$,

$$r([H_m, x_{i+1}]) \supseteq r([H_m, x_i]).$$

Proof: Let i be any integer such that $1 \leq i < m$. $H_m \leq H_i \leq J_i < x_{i+1}$, hence we have the picture:



Hence, $r([H_m, x_{i+1}]) \supseteq r([H_m, H_i]) \cup r([H_i, J_i])$. By 2) c) of Theorem 1, $r([H_i, J_i]) = r([J_i, x_i])$. Hence, $r([H_m, x_{i+1}]) \supseteq r([J_i, x_i])$, and therefore, $r([H_m, x_{i+1}]) \supseteq r([H_m, x_i])$. □

Lemma 5: If $H_m = \min\{H_1, \ldots, H_m\}$, then for all i, $1 \leq i \leq (m-1)$, $r([H_m, x_m]) \supseteq r([J_i, x_{i+1}])$.

Proof: Let i be any integer such that $1 \leq i \leq (m-1)$.

By Lemma 4 $r([H_m, x_m]) \supseteq r([H_m, x_{m-1}])$, $r([H_m, x_{m-1}]) \supseteq r([H_m, x_{m-2}])$, $\ldots$, $r([H_m, x_{i+2}]) \supseteq r([H_m, x_{i+1}])$. Hence, $r([H_m, x_m]) \supseteq r([H_m, x_{i+1}])$. We have $H_m \leq H_i \leq J_i < x_{i+1}$. That is the picture:

Hence, $[H_m, x_{i+1}] \supseteq [J_i, x_{i+1}]$. Hence $r([H_m, x_m]) \supseteq r([J_i, x_{i+1}])$. □

Completion of the Proof of Theorem 1: Without loss of generality we assume $H_m = \min\{H_1, \ldots, H_m\}$. By Lemma 5,

$$r([H_m, x_m]) \supseteq \bigcup_{i=1}^{m-1} r([J_i, x_{i+1}]).$$

By part 2) d) of Theorem 1 we have for some i, $1 \leq i \leq n$, $r([J_m, x_m] \cap I_i = \emptyset$ and $r(x_m) \in U_i$. By part 2) c) of Theorem 1, $r([H_m, x_m]) = r([J_m, x_m])$. Hence,

$$\bigcup_{i=1}^{m-1} r([J_i, x_{i+1}]) \cap I_i = \phi,$$

and

$$\bigcup_{i=1}^{m-1} r([J_i, x_{i+1}]) \cap U_i \neq \phi.$$

Therefore, by (II) $(\bar{r} \mid \pi) \in [\Omega]$.

□

Section 3: Remarks

In [6] Rabin uses the following definition.

Definition: An f.a.t. with designated subsets is a system $\mathcal{A} = \langle S, \Sigma, M, s_0, \mathcal{F} \rangle$, where S is a finite set of states, Σ is a finite set, $M: S \times \Sigma \to P(S \times S)$, and $\mathcal{F} \subseteq P(S)$ is the set of designated subsets. An $\mathcal{A}$-run on $t = (v,T)$ is as defined in Section 2. $\mathcal{A}$ accepts t if there exists an $r \in Rn(\mathcal{A},t)$ such that for all paths $\pi \subset T$, $In(r \mid \pi) \in \mathcal{F}$.

The proof of Theorem 1 can be extended to show that $r([H_m, x_m]) = \bigcup_{i=1}^{m-1} r([J_i, x_{i+1}])$, where H_i, x_i, etc. are as in the proof of Theorem 1.

Hence for $\mathcal{A} = \langle S, \{a\}, M, s_0, \mathcal{F} \rangle$, where $c(S) = q$, we have: $T(\mathcal{A}) \neq \phi \Leftrightarrow$ for some finite tree E there exists an r such that

1) $r \in Rn(\mathcal{A},E)$,

2) there exist mappings $J: Ft(E) \to E\text{-}Ft(E)$ and $H: Ft(E) \to E\text{-}Ft(E)$ such that

   a) $H(x) \leq J(x) < x$,

   b) $r(J(x)) = r(x)$,

   c) $r([H(x), J(x)]) = r([J(x), x])$,

   d) $r([J(x), x]) \in \mathcal{F}$.

The appropriate definition of a good string with respect to $\mathcal{A}$ is a simple modification of the definition of good string used in the proof of Theorem 2. For either definition of good string we can design a nondeterministic finite automaton on finite strings, $\mathfrak{M}$, which recognizes the set of good strings and which has at most $2^{2q}(q+1)$ states. By the subset construction we can design a deterministic automaton $\mathfrak{M}'$ equivalent to $\mathfrak{M}$ such that $\mathfrak{M}'$ has at most $Q = 2^{2^{2q}(q+1)}$ states. Using $\mathfrak{M}'$ we can easily construct a finite automaton on finite trees, $\mathcal{A}'$, such that $T(\mathcal{A}') \neq \phi$ if and only if $T(\mathcal{A}) \neq \phi$ and such that the state set of $\mathcal{A}'$ is the cross product of the state sets of $\mathcal{A}$ and $\mathfrak{M}'$. Hence $\mathcal{A}'$ has at most qQ states. We can determine whether $T(\mathcal{A}') \neq \phi$ in $(q\ Q)^3$ computational steps.

Hence given a finite automaton $\mathcal{A}$ on infinite trees which has q states and uses either notion of acceptance, we can determine whether or not $T(\mathcal{A}) \neq \phi$ in $\left(q\ 2^{2^{2q}(q+1)} \right)^3$ computational steps.

Remark 2: If we have a finite Σ-tree (v,E), and a function $J: Ft(E) \to E\text{-}Ft(E)$ such that for all $x \in Ft(E)$, $v(J(x)) = v(x)$, then we can generate a unique Σ-tree $(\bar{v},T)$ as in the proof of Theorem 1. Call any Σ-tree which can be generated in this way a finitely-generable Σ-tree.

Rabin in [5] defines a Σ-tree, (v,T), to be regular if and only if for each $\sigma \in \Sigma$, $v^{-1}(\sigma)$ is a regular subset of $\{0,1\}^*$. It is easily shown that a Σ-tree is finitely-generable if and only if it is regular.

Remark 3: From Theorem 1 it is easily shown that if an f.a.t. accepts any Σ-tree, then it accepts a finitely-generable Σ-tree. Rabin shows this in [5]. In [2] Buchi and Landweber prove that if $P(X,Y)$ is a finite-state condition and X has a winning strategy, then X has a winning finite-state strategy. Rabin has observed in [5] that the set of winning strategies for X corresponds in a natural way to a set of $\{0,1\}$-trees defined by a (deterministic) infinite tree automaton. Hence, it easily follows from Rabin's result in [5] or from the results in this paper that if X has a winning strategy then X has a winning finite-state strategy.

We can also observe the following. If X does not have a winning strategy, then by our Theorem 1 we see that X does not have a "partial" strategy of a particular kind. From this one can show that Y has a winning strategy for $P(X,Y)$, thus showing that $P(X,Y)$ is determined. This is another result of [2].

CHAPTER 2

The Complementation Problem

Section 1: Introduction

Given an automaton $\mathcal{A}$ on infinite Σ-trees, one defines a Σ-tree t to be accepted by $\mathcal{A}$ if there exists an $\mathcal{A}$-run on t such that for all paths, the sequence of states on that path satisfies a particular property. One can similarly define t to be dually accepted by $\mathcal{A}$ if for all $\mathcal{A}$-runs on t, there exists a path satisfying for that run the particular property. The problem is to construct, for given $\mathcal{A}$, an automaton $\mathcal{A}'$ such that for any valued tree t, t is accepted by $\mathcal{A}'$ if and only if it is not accepted by $\mathcal{A}$. Since given $\mathcal{A}_1$ we can construct an automaton $\mathcal{A}_2$ such that the set of Σ-trees not accepted by $\mathcal{A}_1$ is precisely the set dually accepted by $\mathcal{A}_2$, the complementation problem can be reworded to state that given $\mathcal{A}$, we wish to construct an automaton $\mathcal{A}'$ such that the set of valued trees dually accepted by $\mathcal{A}$ is precisely the set accepted by $\mathcal{A}'$.

That is, we want $\mathcal{A}'$ to accept t if and only if for every $\mathcal{A}$-run on t there is a path such that the sequence of states along it satisfies the property defined by $\mathcal{A}$. The natural thing to look for in constructing $\mathcal{A}'$ is an automaton which can explicitly pick out an appropriate path for each $\mathcal{A}$-run on t. That is, we would like every $\mathcal{A}'$-run on t to specify a path for each $\mathcal{A}$-run on t, and we would like there to be some condition on sequences of $\mathcal{A}'$ states which holds for all paths of the $\mathcal{A}'$-run exactly when for each $\mathcal{A}$-run, the sequence of $\mathcal{A}$ states along the path specified for that run satisfies the $\mathcal{A}$-property.

A natural point of view is to think of starting out at Λ, and having our $\mathcal{A}'$-run choose, for each possible pair of $\mathcal{A}$-states which can

occur at the nodes 0 and 1 (in an $\mathcal{A}$-run on t), whether to continue the path by going left or by going right, and continuing in this way, at each node choosing for each possibility of $\mathcal{A}$-run occurring immediately below it whether to go left or right. An $\mathcal{A}'$-run cannot tell us in one step what path to choose for each $\mathcal{A}$-run. But it can tell us each successive choice of left or right given each successive segment of the $\mathcal{A}$-run.

A state of $\mathcal{A}'$ will be, essentially, a finite sequence of states of $\mathcal{A}$. Each member of an $\mathcal{A}'$ state at a node will represent the last state in the initial segment of a path chosen for the initial segment of a particular $\mathcal{A}$-run. For each member of an $\mathcal{A}'$ state at a node the $\mathcal{A}'$-run must say how to continue the path for each pair of $\mathcal{A}$ transitions possible (for t) beneath that node. It is therefore necessary that a state of $\mathcal{A}'$ actually be a sequence of ordered pairs whose first part is a state of $\mathcal{A}$ and whose second part is a set of backward pointers. Yes, I said a set of backward pointers rather than just one; this is because in reality an ordered pair appearing at a node in an $\mathcal{A}'$-run must represent the last state in the initial segments of (possibly) many paths chosen for the initial segments of (possibly) many $\mathcal{A}$-runs. That is because the sequences making up the set of states of $\mathcal{A}'$ must be bounded in length in order for $\mathcal{A}'$ to be a truly finite automaton. If we insure this bound by insisting that in every state of $\mathcal{A}'$ an $\mathcal{A}$-state can occur at most n times, then we formally denote $\mathcal{A}'$ by $\mathfrak{M}^n_{\mathcal{A}}$.

The fact that everything accepted by $\mathfrak{M}^n_{\mathcal{A}}$ is dually accepted by $\mathcal{A}$ will follow easily from the definition of $\mathfrak{M}^n_{\mathcal{A}}$. The converse, that for sufficiently large n, if t is dually accepted by $\mathcal{A}$ we can find an accepting run for it on $\mathfrak{M}^n_{\mathcal{A}}$, is far from obvious, and our induction proof basically parallels the one Rabin presents in [6]. The difference

between our proof and his can be viewed as being that we keep, at each stage of the induction, information which he discards, so that we arrive at the end with a specific, non-inductive description of the desired automaton.

Section 2: Some Definitions, Theorems, and Proofs

Definition: If S is a set, denote by S^{ω} the set of infinite sequences of members of S. We identify the sequence $s_0, s_1, \ldots$ with the map $\alpha:\omega \to S$ where $\alpha(i)=s_i$.

Notation: If $\langle x,y\rangle$ is an ordered pair, let $p_1(\langle x,y\rangle)=x$ and let $p_2(\langle x,y\rangle)=y$. If $\alpha=\alpha_0,\alpha_1,\ldots$ is an infinite sequence of ordered pairs, let $p_1(\alpha)$=the sequence $p_1(\alpha_0), p_1(\alpha_1), \ldots$. Define $p_2(\alpha)$ similarly. Note that this notation is consistent with thinking of an infinite sequence as a map from ω to a set.

Notation: If $x \in T$ and π is a path of T_x, and $r:T_x \to S$, then denote by $(r|\pi)_{\omega}$ the sequence $r(x_0), r(x_1), \ldots$ where $\pi=\{x_0,x_1,\ldots\}$ and $x_i<x_{i+1}$ for all $i \in \omega$.

Definition: A Generalized Automaton on Trees (G.A.T.) is a system $\mathcal{A}=\langle S,\Sigma,M,s_0,Q\rangle$ where S, Σ, M, s_0, are as in Chapter 1, and Q is a subset of S^{ω}. If t_x is a Σ-tree, let $Rn(\mathcal{A},t_x)$ be defined as in Chapter 1. Define $T(\mathcal{A})$, the set accepted by $\mathcal{A}$, by $T(\mathcal{A})=$

$$\left\{ t_x \;\middle|\; \begin{array}{l} \text{There exists an } r \in Rn(\mathcal{A},t_x) \text{ such that for all paths } \pi \subset T_x,\ (r|\pi)_{\omega} \in Q. \\ \text{(Call r an accepting run of } \mathcal{A} \text{ on } t_x.) \end{array} \right\}$$

Define $D(\mathcal{A})$, the set dually accepted by $\mathcal{A}$, by $D(\mathcal{A})=$ $\{t_x \mid$ For all $r \in Rn(\mathcal{A},t_x)$, there exists a path $\pi \subset T_x$ such that $(r|\pi)_\omega \in Q.\}$

We now define three types of Finite Automata on Trees.

Definition: A pairs-automaton is a system $\mathcal{A}=<S,\Sigma,M,s_0,\Omega>$ where $\Omega=(<L_i,U_i>)_{1\le i\le k}$ is a finite sequence of ordered pairs of subsets of S, and S, Σ, M, s_0 are as above. Let $\mathcal{A}'$ be the G.A.T. $<S,\Sigma,M,s_0,Q>$ where $Q=\{\alpha \in S^\omega \mid \alpha \in [\Omega]\}$. Define $T(\mathcal{A})$ and $D(\mathcal{A})$ to be equal to $T(\mathcal{A}')$ and $D(\mathcal{A}')$ respectively. (Note that a pairs-automaton is the same thing that we meant by an f.a.t. in Chapter 1.) For such an $\mathcal{A}$, we say that Ω is of order k and that $\mathcal{A}$ is of order k. If $L_k=\phi$, then we say that Ω and $\mathcal{A}$ are of order k-empty.

Definition: A sets-automaton is a system $\mathcal{A}=<S,\Sigma,M,s_0,\mathcal{F}>$ where $\mathcal{F} \subseteq P(S)$ (P(S) is the set of subsets of S.) and S, Σ, M, s_0, are as above. Let $\mathcal{A}'$ be the G.A.T. $<S,\Sigma,M,s_0,Q>$ where $Q=\{\alpha \in S^\omega \mid In(\alpha) \in \mathcal{F}\}$. Define $T(\mathcal{A})$ and $D(\mathcal{A})$ to be equal to $T(\mathcal{A}')$ and $D(\mathcal{A}')$ respectively. (Note that a sets-automaton is the same thing that we meant by an automaton with designated subsets in the remarks of Chapter 1.)

Definition: An automaton-automaton is a system $\mathcal{A}=<S,\Sigma,M,s_0,\mathcal{U}>$ where S, Σ, M, s_0, are as above and $\mathcal{U}$ is a deterministic sequential automaton (as defined in [6]) whose inputs are members of S^ω. Let $\mathcal{A}'$ be the G.A.T. $<S,\Sigma,M,s_0,Q>$ where $Q=\{\alpha \in S^\omega \mid \alpha \in T(\mathcal{U})$, the set accepted by $\mathcal{U}$.$\}$ Define $T(\mathcal{A})$ and $D(\mathcal{A})$ to be equal to $T(\mathcal{A}')$ and $D(\mathcal{A}')$ respectively.

These three types of finite automata are all equivalent, in the sense of Facts 1 and 2. These facts are easily shown by Rabin in [6].

Fact 1: Let Σ be a finite set and let W be a set of Σ-trees. Then there exists a pairs-automaton $\mathcal{A}_1$ such that $T(\mathcal{A}_1)=W \Leftrightarrow$ there exists a sets-automaton $\mathcal{A}_2$ such that $T(\mathcal{A}_2)=W \Leftrightarrow$ there exists an automaton-automaton $\mathcal{A}_3$ such that $T(\mathcal{A}_3)=W$.

Fact 2: (same as Fact 1 only with $T(\mathcal{A}_1)$, $T(\mathcal{A}_2)$, $T(\mathcal{A}_3)$ replaced by $D(\mathcal{A}_1)$, $D(\mathcal{A}_2)$, $D(\mathcal{A}_3)$ respectively.)

Now if $\mathcal{A}=\langle S,\Sigma,M,s_0,\mathcal{F}\rangle$ is a sets-automaton, then by its definition we see that $D(\langle S,\Sigma,M,s_0,P(S)-\mathcal{F}\rangle)$ is exactly the complement (with respect to the set of Σ-trees) of $T(\mathcal{A})$. This observation, together with Facts 1 and 2, imply that whichever definition of finite automaton on trees you choose, to solve the complementation problem it is sufficient to exhibit for every pairs-automaton $\mathcal{A}$, an automaton-automaton $\mathcal{A}'$ such that $T(\mathcal{A}')=D(\mathcal{A})$. We shall do this. Now follows the main definition of this chapter.

Notation: If n is a positive integer, let $[n]=\{1,2,\ldots,n\}$.

Definition: Let $\mathcal{A}=\langle S, \Sigma, M, s_0, \Omega\rangle$ be a pairs-automaton. Let n be a positive integer. We will define an automaton-automaton $\mathfrak{M}^n_{\mathcal{A}} = \langle S^n_{\mathcal{A}}, \Sigma, M^n_{\mathcal{A}}, \langle s_0, \phi\rangle, \mathcal{U}^n_{\mathcal{A}}\rangle$. Let $m=c(S)$. Then we define

$$S^n_{\mathcal{A}}=\left\{\alpha \;\middle|\; \begin{array}{l}\alpha \text{ is a finite sequence of members of } S\times P([mn]) \text{ such that for each}\\ s\in S, \text{ there are at most n values of i for which } p_1(\alpha(i))=s.\end{array}\right\}$$

(Every member of P([mn]) is to be thought of as a set of backward pointers. Since each member of S can occur up to n times in a member of $S^n_{\mathcal{A}}$, each member of $S^n_{\mathcal{A}}$ can be as long as mn, hence the need for mn backward pointers.)

The starting state of $\mathfrak{M}^n_{\mathcal{A}}$ is $\langle s_0, \phi\rangle$, a sequence of length 1.

Let $\alpha\in S^n_{\mathcal{A}}$ and let $a\in\Sigma$. Define

$$M^n_{\mathcal{A}}(\alpha,a)=\left\{ <\alpha_0,\alpha_1> \in S^n_{\mathcal{A}} \times S^n_{\mathcal{A}} \;\middle|\; \begin{array}{l} \text{For all } i,\ 1\le i\le \text{length}(\alpha), \text{ if} \\ u=p_1(\alpha(i)) \text{ and if } <u_0,u_1>\in M(u,a), \text{ then there exists} \\ \text{a positive integer } j \text{ such that either} \\ \quad \text{I) } p_1(\alpha_0(j))=u_0 \text{ and } i\in p_2(\alpha_0(j)) \quad \text{or} \\ \quad \text{II) } p_1(\alpha_1(j))=u_1 \text{ and } i\in p_2(\alpha_1(j)). \end{array} \right\}$$

(Think of $\alpha(i)$ as representing the value at some node x of a run of $\mathcal{A}$ on (v,T). If $p_1(\alpha(i))=u$ and $<u_0,u_1>\in M(u,v(x))$, then each $\mathfrak{M}^n_{\mathcal{A}}$-run must tell us either to go left to an element whose first part is u_0 or right to an element whose first part is u_1.)

It remains to define $\mathcal{U}^n_{\mathcal{A}}$, and we will do this by defining a set $Q\subseteq (S^n_{\mathcal{A}})^\omega$ and then arguing that there exists a deterministic sequential automaton which accepts exactly Q. Firstly, if $\alpha\in (S^n_{\mathcal{A}})^\omega$, $\alpha=\alpha_0,\alpha_1,\ldots$, define a <u>thread of α</u> to be an infinite sequence of integers, $J=j_0,j_1,\ldots$ such that for all $i\in\omega$, $\alpha_i(j_i)$ is defined, and such that for all $i\in\omega$, $j_i\in p_2(\alpha_{i+1}(j_{i+1}))$. Define the <u>S-sequence associated with the thread J (for α)</u> to be the sequence $\beta=p_1(\alpha_0(j_0)),p_1(\alpha_1(j_1)),\ldots$ so that $\beta\in S^\omega$. Define

$Q=\{\alpha\in (S^n_{\mathcal{A}})^\omega \mid$ For every S-sequence β associated with a thread of α, $\beta\in[\Omega].\}$.

So we have that $\bar{Q}=(S^n_{\mathcal{A}})^\omega - Q=$

$\{\alpha\in (S^n_{\mathcal{A}})^\omega \mid$ for <u>some</u> thread of α, the associated S-sequence is not of type $[\Omega]\}$.

It is easy to see that there exists a <u>nondeterministic</u> sequential automaton, $\mathcal{U}_1$, such that $T(\mathcal{U}_1)=\bar{Q}$. But McNaughton [4] has shown that for every nondeterministic sequential automaton $\mathcal{U}_1$ one can construct a deterministic one, $\mathcal{U}_2$, such that $T(\mathcal{U}_2)=T(\mathcal{U}_1)$. But it is easy to see that given any such $\mathcal{U}_2$ there exists a deterministic sequential automaton, $\mathcal{U}_3$, which accepts precisely the complement of $T(\mathcal{U}_2)$. So

let $\mathcal{U}^n_{\mathcal{A}}$ be such that $T(\mathcal{U}^n_{\mathcal{A}})=Q$. This finally completes the definition of $\mathfrak{M}^n_{\mathcal{A}}$.

Notation: Very often, if R is a $\mathfrak{M}^n_{\mathcal{A}}$-run, we will want to refer to the i^{th} element of R(x) for some node x and some positive integer i. So instead of writing (R(x))(i) we will write R(x,i).

So if $t_x=(v,T_x)$ is a Σ-tree , $R\in Rn(\mathfrak{M}^n_{\mathcal{A}},t_x)$, $y\in T_x$, $u=p_1(R(y,i))$, and $\langle u_0,u_1\rangle\in M(u,v(y))$, then we require that there exist $\sigma\in\{0,1\}$ and a positive integer j such that $p_1(R(y\cdot\sigma,j))=u_\sigma$ and $i\in p_2(R(y\cdot\sigma,j))$.

Remark: Let $\mathcal{A}$ be a Σ-pairs-automaton, and let n_1 and n_2 be positive integers with $n_1\leq n_2$. Then since $S^{n_1}_{\mathcal{A}}\subseteq S^{n_2}_{\mathcal{A}}$, and if $\langle\alpha,a\rangle\in S^{n_1}_{\mathcal{A}}\times\Sigma$ then $M^{n_1}_{\mathcal{A}}(\alpha,a)\subseteq M^{n_2}_{\mathcal{A}}(\alpha,a)$, and $T(\mathcal{U}^{n_1}_{\mathcal{A}})\subseteq T(\mathcal{U}^{n_2}_{\mathcal{A}})$, we have that $T(\mathfrak{M}^{n_1}_{\mathcal{A}})\subseteq T(\mathfrak{M}^{n_2}_{\mathcal{A}})$.

We now state the main theorem of this chapter.

Theorem 3: Let $\mathcal{A}$ be a pairs-automaton of order k. Let

$n=\sum_{i=0}^{k} 2^{\binom{i}{2}}$. Then $D(\mathcal{A})=T(\mathfrak{M}^n_{\mathcal{A}})$.

Half of Theorem 3 follows from Lemma 6.

Lemma 6: Let $\mathcal{A}=\langle S,\Sigma,M,s_0,\Omega\rangle$ be a pairs-automaton. Let n be a positive integer. Then $D(\mathcal{A})\supseteq T(\mathfrak{M}^n_{\mathcal{A}})$.

Proof of Lemma 6: Let t=(v,T) be a Σ-tree such that $t\in T(\mathfrak{M}^n_{\mathcal{A}})$. (Since it is clear that for any of the tree automata we have defined, (v_0,T_x) is accepted (dually accepted) if and only if (v'_0,T) is accepted (dually accepted) where $v'_0(y)=v_0(x\cdot y)$ for $y\in T$, to prove Lemma 6 it is sufficient to show that $t\in D(\mathcal{A})$.)

Let R be an accepting run of $\mathfrak{M}^n_{\mathcal{A}}$ on t, that is, $R \in Rn(\mathfrak{M}^n_{\mathcal{A}}, t)$ and for all paths $\pi \subset T$, $(R|\pi)_\omega \in T(\mathcal{U}^n_{\mathcal{A}})$. Let $r \in Rn(\mathcal{A}, t)$. We wish to find a path $\pi \subset T$ such that $(r|\pi)_\omega \in [\Omega]$. To do this, we define by induction a function $f:\omega \to T$ such that $f(0)=\Lambda$ and $f(i+1)$=either $f(i)\cdot 0$ or $f(i)\cdot 1$, and $rf \in [\Omega]$. Then we merely let $\pi=\{f(0),f(1),. . .\}$.

Simultaneously with f we define a function $g:\omega \to [mn]$ where $m=c(S)$. (The idea is that the i^{th} element of the path we construct will correspond to $R(f(i),g(i))$.) As an induction hypothesis at stage i we assume: $r(f(i))=p_1(R(f(i),g(i)))$ and (for $i>0$) $g(i-1) \in p_2(R(f(i),g(i)))$.

Define $f(0)=\Lambda$ and $g(0)=1$. Clearly the induction hypothesis holds so far. Assume that f(i) and g(i) have been defined and that the induction hypothesis is true for i. We will define f and g at i+1. Let $x=f(i)$. Then $r(x)=p_1(R(x,g(i)))$. Let $u_0=r(x\cdot 0)$ and $u_1=r(x\cdot 1)$. Then $<u_0,u_1> \in M(r(x),v(x))$. By the definition of $\mathfrak{M}^n_{\mathcal{A}}$ run we have that there is a positive integer j and a $\sigma \in \{0,1\}$ such that $u_\sigma=p_1(R(x\cdot\sigma,j))$ and $g(i) \in p_2(R(x\cdot\sigma,j))$. Define $f(i+1)=x\cdot\sigma$ and $g(i+1)=j$. Clearly the induction hypothesis holds at i+1.

Let $\pi=\{f(0),f(1),. . .\}$. π is a path of T so, since R is accepting, $(R|\pi)_\omega \in T(\mathcal{U}^n_{\mathcal{A}})$. By the hypothesis we have carried through the definition of f and g, we see that the sequence g(0),g(1),. . . is a thread of $(R|\pi)_\omega$ and that $(r|\pi)_\omega =$

$p_1(R(f(0),g(0)))$, $p_1(R(f(1),g(1)))$, $p_1(R(f(2),g(2)))$,. . . =

the S-sequence associated with that thread. By the definition of $\mathcal{U}^n_{\mathcal{A}}$ we see that $(r|\pi)_\omega \in [\Omega]$. □

The other half of Theorem 3, namely that for $\mathcal{A}$ and n as in Theorem 3 $D(\mathcal{A}) \subseteq T(\mathfrak{M}^n_{\mathcal{A}})$, follows trivially, by induction, from the next three Theorems.

Theorem 4: Let $\mathcal{A}$ be a pairs-automaton of order 0 (that is, the sequence of pairs is empty). Then $D(\mathcal{A}) \subseteq T(\mathfrak{M}^1_{\mathcal{A}})$.

Theorem 5: Let k be a nonnegative integer and let n be a positive integer such that for every pairs-automaton $\mathcal{B}$ of order k, $D(\mathcal{B}) \subseteq T(\mathfrak{M}^n_{\mathcal{B}})$. Then for every pairs-automaton $\mathcal{A}$ of order k+1-empty, $D(\mathcal{A}) \subseteq T(\mathfrak{M}^n_{\mathcal{A}})$.

Theorem 6: Let k and n be positive integers such that for every pairs-automaton $\mathcal{B}$ of order k-empty, $D(\mathcal{B}) \subseteq T(\mathfrak{M}^n_{\mathcal{B}})$. Then for every pairs-automaton $\mathcal{A}$ of order k, $D(\mathcal{A}) \subseteq T(\mathfrak{M}^{n2^k}_{\mathcal{A}})$.

Before we prove Theorem 4, we need the following definition.

Definition: Let $\mathcal{A} = <S, \Sigma, M, s_0, \Omega>$ be a pairs-automaton. Let $s \in S$. Then define $\mathcal{A}_s$ to be the pairs-automaton $\mathcal{A}_s = <S, \Sigma, M, s, \Omega>$. That is, $\mathcal{A}_s$ is the same as $\mathcal{A}$ except that the initial state is changed to s.

Proof of Theorem 4:

Let $\mathcal{A} = <S, \Sigma, M, s_0, \Omega>$ be a pairs-automaton with Ω=the empty sequence and c(S)=m. Let t=(v,T) be a Σ-tree such that $t \in D(\mathcal{A})$. We wish to show that $t \in T(\mathfrak{M}^1_{\mathcal{A}})$. (Recall that no map is of type Ω.)

Since $t \in D(\mathcal{A})$, there are no $\mathcal{A}$-runs on t, that is, $Rn(\mathcal{A}, t) = \phi$. For n a nonnegative integer and for $x \in T$ such that length(x) $\leq$ n, define the finite tree $E^n_x = \{y \in T_x \mid \text{length}(y) \leq n\}$. Define $e^n_x = (v, E^n_x)$. By an application of König's Lemma we see that for some n, $Rn(\mathcal{A}, e^n_\Lambda) = \phi$. Let N be such a number, that is, $Rn(\ , e^N_\Lambda) = \phi$.

Define a map $R: T \to S^1$ as follows: $R(\Lambda) = <s_0, \phi>$; for $x \in E^N_\Lambda$, let $R(x) = <s_1, [m]>, <s_2, [m]>, \ldots, <s_\ell, [m]>$ where $s_1, s_2, \ldots, s_\ell$ is some enumeration of the set $\{s \in S \mid Rn(\mathcal{A}_s, e^N_x) = \phi\}$; for x such that

length(x) > N, let R(x)=the empty sequence.

Clearly if π is a path of T, $(R|\pi)_\omega$ has no threads. Hence, if R is a run of $\mathfrak{M}^1_{\mathcal{A}}$ on t, it is an accepting run. So let $x \in E^N_\Lambda$ and let $s=p_1(R(x,i))$ for some i. Then $Rn(\mathcal{A}_s, e^N_x)=\phi$. Let $\langle u_0,u_1\rangle \in M(s,v(x))$. Then for some $\sigma \in \{0,1\}$, $Rn(\mathcal{A}_{u_\sigma}, e^N_{x\cdot\sigma})=\phi$. So there exists a j such that $u_\sigma=p_1(R(x\cdot\sigma,j))$ and $i \in p_2(R(x\cdot\sigma,j))$. So R is a run on t. So $t \in T(\mathfrak{M}^1_{\mathcal{A}})$. □

Theorems 5 and 6 will be proved in the next two sections. To this end, we make some more definitions.

Definition: Let $\mathcal{A}=\langle S, \Sigma, M, s_0, \Omega\rangle$ be a pairs-automaton. Denote by $\bar{\mathcal{A}}$ the $\Sigma \times P(S)$ automaton $\langle S, \Sigma \times P(S), \bar{M}, s_0, \Omega\rangle$ where $\bar{M}$ is defined as follows. Let $s_1 \in S$, $a \in \Sigma$, $S_1 \subseteq S$. Define

$$\bar{M}(s_1,\langle a,S_1\rangle)=\begin{cases} M(s_1,a) \text{ if } s_1 \notin S_1 \\ \phi \text{ if } s_1 \in S_1 \end{cases}.$$

Definition: Let $t_x=(v,T_x)$ be a Σ-tree, let S be a set, and let H be a subset of $T \times S$. Then define t_xH to be the $\Sigma \times P(S)$-tree, $(\bar{v},T_x)$, where $\bar{v}(y)=\langle v(y),\{s \in S \mid \langle y,s\rangle \in H\}\rangle$ for $y \in T_x$.

Intuitive Remarks: Let $\mathcal{A}=\langle S, \Sigma, M, s_0, \Omega\rangle$ be a pairs-automaton. Let $t=(v,T)$ be a Σ-tree. Let $H \subseteq T \times S$.

Every run of $\bar{\mathcal{A}}$ on tH is also a run of $\mathcal{A}$ on t, but there might be fewer of the former since $Rn(\bar{\mathcal{A}},tH)$ is precisely those members, r, of $Rn(\mathcal{A},t)$ which don't "run into" a member of H, that is, $r \in Rn(\bar{\mathcal{A}},tH)$ if and only if $\langle y,r(y)\rangle \notin H$ for all $y \in T$. Say that $t \in D(\mathcal{A})$. Then $tH \in D(\bar{\mathcal{A}})$. If our goal is to show that $t \in T(\mathfrak{M}^n_{\mathcal{A}})$, it might be easier to show first that $tH \in T(\mathfrak{M}^n_{\bar{\mathcal{A}}})$ since $Rn(\bar{\mathcal{A}},tH)$ might in some sense be a simpler set that $Rn(\mathcal{A},t)$.

Consider what a run of $\mathfrak{M}^n_{\mathcal{A}}$ on tH looks like. It starts out the same as a run of $\mathfrak{M}^n_{\mathcal{A}}$ on t, that is it's a map $R:T \to S^n_{\mathcal{A}}$ such that $R(\Lambda)=<s_0, \phi>$, and it continues like a run of $\mathfrak{M}^n_{\mathcal{A}}$ on t except that certain threads are allowed to die at members of H. For example, say that for some x and i, $p_1(R(x,i))=u$ and $<u_0,u_1> \in M(u,v(x))$. If R were a run of $\mathfrak{M}^n_{\mathcal{A}}$ on t, we would require there to exist a j such that for some $\sigma \in \{0,1\}$, $u_\sigma = p_1(R(x \cdot \sigma, j))$ and $i \in p_2(R(x \cdot \sigma, j))$. Since R is only a $\mathfrak{M}^n_{\mathcal{A}}$ -run on tH, we require this condition only if $<x,s> \notin H$. Hence, for a path $\pi \subset T$, $(R|\pi)_\omega$ may have fewer threads than if $R \in Rn(\mathfrak{M}^n_{\mathcal{A}}, t)$ and therefore, as already claimed, it might be easier to find an accepting run of $\mathfrak{M}^n_{\mathcal{A}}$ on tH than of $\mathfrak{M}^n_{\mathcal{A}}$ on t.

How do these observations lead us towards constructing an accepting $\mathfrak{M}^n_{\mathcal{A}}$-run on t? Maybe we can find a set $H_0 \subseteq T \times S$ such that we can show $tH_0 \in T(\mathfrak{M}^n_{\mathcal{A}})$. Let R_0 be an accepting run of $\mathfrak{M}^n_{\mathcal{A}}$ on tH_0. R_0 is partly an accepting run of $\mathfrak{M}^n_{\mathcal{A}}$ on t, except that if $<x,s> \in H_0$ and $p_1(R_0(x,i))=s$, then R_0 doesn't continue properly for the i^{th} element of $R_0(x)$. But, maybe we can find a set $H_{<x,s>}$ such that we can show that $t_x H_{<x,s>} \in T(\mathfrak{M}^n_{\mathcal{A}_s})$, that is, there exists an accepting run $R_{<x,s>}$ of $\mathfrak{M}^n_{\mathcal{A}_s}$ on $t_x H_{<x,s>}$. Then we can use $R_{<x,s>}$ to continue threads of R_0 which had died. But for $<x',s'> \in H_{<x,s>}$ we need a set $H_{<x',s'>}$, etc. For the sake of uniformity we will refer to H_0 as $H_{<\Lambda,s_0>}$.

It is important to understand what $R_{<x,s>} \in Rn(\mathfrak{M}^n_{\mathcal{A}_s}, t_x H_{<x,s>})$ means. $R_{<x,s>}$ differs from a member of $Rn(\mathfrak{M}^n_{\mathcal{A}}, tH_{<x,s>})$ only by starting at x instead of Λ and by starting with $<s, \phi>$ rather than $<s_0, \phi>$. Assume now that we have a set $H \subseteq T \times S$ such that $<\Lambda, s_0> \in H$

and such that for every $\langle x,s\rangle \in H$ we have a subset of H, $H_{\langle x,s\rangle}$, with $H_{\langle x,s\rangle} \subseteq (T-\{x\}) \times S$. Assume furthermore that for every $\langle x,s\rangle \in H$ we have an accepting run $R_{\langle x,s\rangle} \in Rn(\mathfrak{M}^n_{\sigma\tau_s}, t_x H_{\langle x,s\rangle})$. What we would like to do is "put together" the runs $R_{\langle x,s\rangle}$ to form an accepting run R of $\mathfrak{M}^n_{\sigma\tau}$ on t, as in the above paragraph.

So for every x and every i, $1 \le i \le \text{length}(R(x))$, R(x,i) will be associated with some $\langle y,u\rangle \in H$ and some integer j such that $y \le x$ and $p_1(R(x,i)) = p_1(R_{\langle y,u\rangle}(x,j))$. We use this association to determine which elements of $R(x^\frown 0)$ and $R(x^\frown 1)$ should point backwards to R(x,i).

The problem that remains is how to decide what members of H the elements of $R(x^\frown 0)$ and $R(x^\frown 1)$ should be associated with. These decisions will be made so that R has the property that for any path $\pi \subset T$ and any thread of $(R|\pi)_\omega$, if the members of that thread are altogether associated with only a finite number of members of H, then that thread eventually has the same S-sequence as a thread of $(R_{\langle y,u\rangle}|\pi)_\omega$ (for some $\langle y,u\rangle \in H$), which by hypothesis is of type Ω. The decisions must also be made in such a way that for any path $\pi \subset T$ and for any thread of $(R|\pi)_\omega$, if the members of that thread are associated with infinitely many members of H then the S-sequence of that thread is of type Ω.

To prove both Theorems 5 and 6 we will find H, $\{H_{\langle x,s\rangle}\}$, and $\{R_{\langle x,s\rangle}\}$ as above. However the way they are obtained and the way the decisions referred to above are made will be different in the two proofs.

In order to combine $\mathfrak{M}^n_{\sigma\tau}$-runs it will be convient to have a notion

of such a run being well-formed.

Definition: Let $\mathcal{A}=<S, \Sigma, M, s_0, \Omega>$ be a pairs-automaton, with $S=\{s_0,s_1,\ldots,s_{m-1}\}$, that is, we have imposed an ordering on S. Let t_x be a Σ-tree. Let n be a positive integer and let $R \in Rn(\mathfrak{M}^n_{\mathcal{A}}, t_x)$. Then we say that R is Well-formed (with respect to the ordering on S, although this will usually not be explicitly stated in the future) if for all $y>x$, length(R(y))=mn, and if for all i, $0 \leq i \leq m-1$, and all j, $1 \leq j \leq n$, it is the case that $p_1(R(y,ni+j))=s_i$. That is, R(y), for $y>x$, is of the form:

$<s_0,A^y_{0,1}>,<s_0,A^y_{0,2}>,\ldots,<s_0,A^y_{0,n}>,<s_1,A^y_{1,1}>,\ldots,<s_1,A^y_{1,n}>,\ldots,<s_{m-1},A^y_{m-1,n}>$

where each $A^y_{i,j} \subseteq [mn]$.

Lemma 7: Let $\mathcal{A}=<S, \Sigma, M, s_0, \Omega>$ be a pairs-automaton where $S=\{s_0,s_1,\ldots,s_{m-1}\}$. Let $t_x=(v,T_x)$ be a Σ-tree. Let n be a positive integer. Let R be an accepting run of $\mathfrak{M}^n_{\mathcal{A}}$ on t_x. Then there exists an accepting run, R', of $\mathfrak{M}^n_{\mathcal{A}}$ on t_x which is well-formed.

Proof of Lemma 7: The proof of Lemma 7 is actually quite easy in concept, but we shall do it in detail anyway.

Let $\mathcal{A}$, t_x, and R be as in the lemma statement. Choose $f:\{<x,1>\} \cup ((T_x-\{x\}) \times [mn]) \rightarrow [mn]$ such that

a) if $0 \leq i \leq m-1$ and $1 \leq j \leq n$, and if $y \in T_x$ and f(y,ni+j) and R(y,f(y,ni+j)) are defined, then $p_1(R(y,f(y,ni+j)))=s_i$ and

b) for every $y \in T_x$ and every ℓ, $1 \leq \ell \leq$ length(R(y)), there exists ℓ' such that $f(y,\ell')=\ell$.

Define R' as follows. Let $R'(x)=<s_0,\phi>$. For $z \in T_x$, $\sigma \in \{0,1\}$, and $y=z^\frown\sigma$, let R'(y) be the string of length mn such that for $0 \leq i \leq m-1$ and $1 \leq j \leq n$ and $q=ni+j$, we have

c) $p_1(R'(y,q))=s_i$ and

d) if $f(y,q) > \text{length}(R(y))$, then $p_2(R'(y,q))=$
$\{\ell \in [mn] \mid f(z,\ell)$ is defined and $f(z,\ell) > \text{length}(R(z))\}$

e) if $f(y,q) \leq \text{length}(R(y))$, then $p_2(R'(y,q))=\{\ell \in [mn] \mid f(z,\ell)$ is defined, and either $f(z,\ell) > \text{length}(R(z))$ or $f(z,\ell) \in p_2(R(y,f(y,q)))\}$

R' is well-formed. To show that $R' \in Rn(\mathfrak{M}^n_\tau, t_x)$ let $z \in T_x$, let ℓ be such that $R'(z,\ell)$ is defined, let $u=p_1(R'(z,\ell))$ and let $<u_0,u_1> \in M(u,v(z))$. If $f(z,\ell) > \text{length}(R(z))$ then by c), d), and e)we see that for $\sigma \in \{0,1\}$ and some q, $p_1(R'(z^\frown\sigma,q))=u_\sigma$ and $\ell \in p_2(R'(z^\frown\sigma,q))$. If $f(z,\ell) \leq \text{length}(R(z))$, then by a) and c), $p_1(R(z,f(z,\ell)))=u$. So for some $\sigma \in \{0,1\}$ and some q , $p_1(R(z^\frown\sigma,q))=u_\sigma$ and $f(z,\ell) \in p_2(R(z^\frown\sigma,q))$. By b), there exists q' such that $f(z^\frown\sigma,q')=q$. By a) and c) $p_1(R'(z^\frown\sigma,q'))=p_1(R(z^\frown\sigma,q))=u_\sigma$. Also, since $f(z,\ell) \in p_2(R(z^\frown\sigma,f(z^\frown\sigma,q')))$, we have by e) that $\ell \in p_2(R'(z^\frown\sigma,q'))$. So $R' \in Rn(\mathfrak{M}^n_\alpha, t_x)$.

It remains to show that R' is accepting. So let $\pi \subset T_x$ be a path, $\pi=\{x_0,x_1,\ldots\}$ where $x_i < x_{i+1}$ for $i \in \omega$. Let $\alpha=\ell_0,\ell_1,\ldots$ be a thread of$(R'|\pi)_\omega$. Note that $f(x_0,\ell_0)=f(x,1)=1$ so that $f(x_0,\ell_0) \leq \text{length}(R(x_0))$. Now let $i \in \omega$ be such that $f(x_i,\ell_i) \leq \text{length}(R(x_i))$. So, since $\ell_i \in p_2(R'(x_{i+1},\ell_{i+1}))$, we have by d) and e) above that $f(x_{i+1},\ell_{i+1}) \leq \text{length}(R(x_{i+1}))$ and $f(x_i,\ell_i) \in p_2(R(x_{i+1},f(x_{i+1},\ell_{i+1})))$.

So by induction we see that the infinite sequence $\beta=f(x_0,\ell_0),f(x_1,\ell_1),\ldots$ is a thread of $(R|\pi)_\omega$. But by a) and c), the S-sequence associated with α for $(R'|\pi)_\omega$ is the same as that associated with β for $(R|\pi)_\omega$, and this is of type Ω since R is accepting. □

Section 3: Proof of Theorem 5

Let k be a nonnegative integer and let n be a positive integer such that for every pairs-automaton $\mathcal{A}$ of order k, $D(\mathcal{A}) \subseteq T(\mathfrak{M}^n_{\mathcal{A}})$. Let $\mathcal{A} = \langle S, \Sigma, M, s_0, \Omega \rangle$ be a pairs-automaton where $\Omega = (\langle L_i, U_i \rangle)_{1 \le i \le k+1}$, $L_{k+1} = \phi$, and $S = \{s_0, s_1, \ldots, s_{m-1}\}$. Let $t = (v, T)$ be a fixed Σ-tree such that $t \in D(\mathcal{A})$. We wish to show, eventually, that $t \in T(\mathfrak{M}^n_{\mathcal{A}})$.

Define a set $H \subseteq T \times S$ by $H = \{\langle \Lambda, s_0 \rangle\} \cup \{\langle x, s \rangle \mid t_x \in D(\mathcal{A}_s) \text{ and } s \in U_{k+1}\}$. For $\langle x, s \rangle \in H$, define $H_{\langle x,s \rangle} = \{\langle y, u \rangle \in H \mid y > x\}$. Let $\Omega' = (\langle L_i, U_i \rangle)_{1 \le i \le k}$.

Lemma 8: For every $\langle x, s \rangle \in H$ and for every $r \in Rn(\mathcal{A}_s, t_x)$, either

I) there exists a path $\pi \subset T_x$ such that $r|\pi \in [\Omega']$ or

II) there exists $y > x$ such that $\langle y, r(y) \rangle \in H_{\langle x,s \rangle}$.

Proof of Lemma 8: Assume otherwise. Let $\langle x, s \rangle \in H$ and let $r \in Rn(\mathcal{A}_s, t_x)$ be such that for all paths $\pi \subset T_x$, $r|\pi \notin [\Omega']$, and such that for all $y > x$, $\langle y, r(y) \rangle \notin H$.

Definition: If $Y \subset T_x$ is a set of (pairwise) incomparable nodes (under $\le$) define $T_x^Y = \{z \in T_x \mid \text{for all } y \in Y,\ z \not\ge y\} = T_x - \bigcup_{y \in Y}(T_y - \{y\})$.

We will now define $r' \in Rn(\mathcal{A}_s, t_x)$ such that for all paths $\pi \subset T_x$, $r'|\pi \notin [\Omega]$, contradicting the fact that $t_x \in D(\mathcal{A}_s)$. Let $Y = \{y \in T_x \mid y > x,\ r(y) \in U_{k+1}, \text{ and for all } z,\ x < z < y,\ r(z) \notin U_{k+1}\}$. Clearly the members of Y are incomparable. If $z \in T_x^Y$, define $r'(z) = r(z)$.

Now let $y \in Y$, $r(y) = u$. Since $u \in U_{k+1}$ and $\langle y, u \rangle \notin H$, it must be the case that $t_y \notin D(\mathcal{A}_u)$. So let $r_y \in Rn(\mathcal{A}_u, t_y)$ be such that for all paths $\pi \subset T_y$, $r_y|\pi \notin [\Omega]$. For all $y \in Y$ and all $z \in T_y$, define $r'(z) = r_y(z)$.

Clearly r' is a run of $\mathcal{A}_s$ on t_x. Let $\pi \subset T_x$ be a path.

Case 1: $\pi \cap Y = \phi$

Then $r'|\pi = r|\pi$ so $r'|\pi \notin [\Omega']$. But $r'(\pi - \{x\}) = r(\pi - \{x\})$ must not intersect U_{k+1}. So $r'|\pi \notin [\Omega]$.

Case 2: $\pi \cap Y \neq \phi$

Let $\{y\} = \pi \cap Y$. Let $\pi_y = \pi \cap T_y$. Then $r'|\pi_y = r_y|\pi_y$. So $r'|\pi_y \notin [\Omega]$. So $r'|\pi \notin [\Omega]$.

Contradiction. □

Now define the pairs -automaton $\mathcal{B} = \langle S, \Sigma, M, s_0, \Omega' \rangle$. What Lemma 8 says is that for $\langle x,s \rangle \in H$, $t_x H_{\langle x,s \rangle} \in D(\overline{\mathcal{T}}_s)$. But $\overline{\mathcal{T}}_s$ is of order k. So by the hypothesis, for $\langle x,s \rangle \in H$, $t_x H_{\langle x,s \rangle} \in T(\mathfrak{M}^n_{\overline{\mathcal{T}}_s})$. So for $\langle x,s \rangle \in H$, let $R_{\langle x,s \rangle}$ be an accepting run of $\mathfrak{M}^n_{\overline{\mathcal{T}}_s}$ on $t_x H_{\langle x,s \rangle}$. But Ω' is a subsequence of Ω, so each $R_{\langle x,s \rangle}$ is also an accepting run of $\mathfrak{M}^n_{\overline{\mathcal{T}}_s}$ on $t_x H_{\langle x,s \rangle}$.

Without loss of generality (by Lemma 7) assume that for $\langle x,s \rangle \in H$, $R_{\langle x,s \rangle}$ is a well-formed accepting run of $\mathfrak{M}^n_{\overline{\mathcal{T}}_s}$ on $t_x H_{\langle x,s \rangle}$. We want to construct a well-formed accepting run, R, of $\mathfrak{M}^n_{\mathcal{T}}$ on t. $R(\Lambda)$ will equal $\langle s_0, \phi \rangle$ and for $x > \Lambda$, R(x) will be of length mn. Simultaneously with R we will construct a function f which is defined at a node x and integer ℓ if $1 \leq \ell \leq \text{length}(R(x))$. If defined, $f(x,\ell)$ will be a member of H, say for example $\langle y,u \rangle$. The interpretation is that we think of the ℓ^{th} element of R(x) being continued like some element of $R_{\langle y,u \rangle}(x)$. Like which element? Well, if y=x, then $R_{\langle y,u \rangle}(x)$ has only one element, namely, $\langle u, \phi \rangle$, and we better make sure that $p_1(R(x,\ell)) = u$. Otherwise, that is if $y > x$, we associate $R(x,\ell)$ with $R_{\langle y,u \rangle}(x,\ell)$, which we can do since by the definition of well-formed both elements have the

same first part. To speak of the two cases uniformly we define a function g such that we always think of associating $R(x,\ell)$ with $R_{f(x,\ell)}(x,g(x,\ell))$. We also have to make sure that $R_{f(x,\ell)}$ gives us acceptable directions on how to continue the ℓ^{th} element of R(x), and therefore carry along various induction hypotheses in the definition of R and f. To define f, we will need a well-ordering on H.

We now begin more formally. Let $\preccurlyeq$ be a fixed well-ordering on $T \times S$; clearly $\preccurlyeq$ induces a well-ordering on H. Denote by $\prec$ the usual strict well-ordering determined by $\preccurlyeq$.

We now define $R:T \to S^n_\omega$ and $f:\{<\Lambda,1>\} \cup ((T-\{\Lambda\}) \times [mn]) \to H$. We will define them by induction, and at every stage the following will be true:

If R(x) is defined and $1 \leq \ell \leq \text{length}(R(x))$, then $f(x,\ell)$ is defined; if $f(x,\ell)=<y,u>$ and $p_1(R(x,\ell))=s$, then

a) $y \leq x$ and

b) if $y=x$ then $u=s$ and

c) $<x,s> \notin H_{<y,u>}$.

Let $R(\Lambda)=<s_0,\emptyset>$. Let $f(\Lambda,1)=<\Lambda,s_0>$. Clearly the above hypothesis holds so far.

Assume now that R(x) and f have been defined so that the above hypothesis holds. For ℓ, $1 \leq \ell \leq \text{length}(R(x))$, define

$$g(x,\ell)=\begin{cases} 1 \text{ if } p_1(f(x,\ell))=x \\ \ell \text{ if } p_1(f(x,\ell)) \neq x \end{cases} .$$

Let $\sigma \in \{0,1\}$. Define $R(x^\frown\sigma)$ to be of length mn such that for $1 \leq q \leq mn$, $p_1(R(x^\frown\sigma,q))=s_i$ where i= the greatest integer less that $\frac{q}{n}$, and such that

$p_2(R(x^\frown\sigma,q))=\{\ell \mid f(x,\ell)$ is defined and $g(x,\ell)\in p_2(R_{f(x,\ell)}(x^\frown\sigma,q))\}$.

We now define f. Let $s=p_1(R(x^\frown\sigma,q))$. If $\langle x^\frown\sigma,s\rangle\in H$ let $f(x^\frown\sigma,q)=\langle x^\frown\sigma,s\rangle$. Clearly this preserves the induction hypothesis. If $\langle x^\frown\sigma,s\rangle\notin H$, let $f(x^\frown\sigma,q)=\underset{\text{under }\preccurlyeq}{\text{minimum}}\{f(x,\ell)\mid \ell\in p_2(R(x^\frown\sigma,q))\}$. Clearly a) and b) are preserved for $x^\frown\sigma$, and since $\langle x^\frown\sigma,s\rangle\notin H$, c) is true at $x^\frown\sigma$.

It remains to show that $R\in Rn(\mathfrak{M}^n_\alpha,t)$, and that R is accepting. Let x,ℓ,s,u_0,u_1, be such that $R(x,\ell)$ is defined, $p_1(R(x,\ell))=s$, and $\langle u_0,u_1\rangle\in M(s,v(x))$. Either by b) or by the fact that R and $R_{f(x,\ell)}$ are well-formed (depending on whether or not $p_1(f(x,\ell))=x$) we can conclude that $p_1(R_{f(x,\ell)}(x,g(x,\ell)))=s$. Now recall that if $f(x,\ell)=\langle y,u\rangle$, then $R_{f(x,\ell)}$ is a run of $\mathfrak{M}^n_{\sigma_u}$ on $t_yH_{\langle y,u\rangle}$. Since $\langle x,s\rangle\notin H_{\langle y,u\rangle}$ (by c)) we know that there exists $\sigma\in\{0,1\}$ and q such that $p_1(R_{f(x,\ell)}(x^\frown\sigma,q))=u_\sigma$ and $g(x,\ell)\in p_2(R_{f(x,\ell)}(x^\frown\sigma,q))$. Since R and $R_{f(x,\ell)}$ are well-formed, $p_1(R(x^\frown\sigma,q))=u_\sigma$. And by the definition of R, $\ell\in p_2(R(x^\frown\sigma,q))$. So $R\in Rn(\mathfrak{M}^n_\alpha,t)$.

To show that R is accepting, let $\pi\subset T$ be a path, $\pi=\{x_0,x_1,\ldots\}$ where $x_i<x_{i+1}$ for $i\in\omega$, and let $\alpha=\ell_0,\ell_1,\ldots$ be a thread of $(R\mid\pi)_\omega$.

Case 1: For infinitely many i, $x_i=p_1(f(x_i,\ell_i))$.

By b), $p_1(R(x_i,\ell_i))=p_2(f(x_i,\ell_i))$ for infinitely many i. But by the definition of H, this implies that $p_1(R(x_i,\ell_i))\in U_{k+1}$ for infinitely many i, and hence, that the S-sequence associated with α is of type Ω.

Case 2: $x_i=p_1(f(x_i,\ell_i))$ for only finitely many i.

So for sufficiently large j, $x_{j+1}\neq p_1(f(x_{j+1},\ell_{j+1}))$. By the definition of f, we have that for sufficiently large j $f(x_{j+1},\ell_{j+1})\preccurlyeq f(x_j,\ell_j)$. Since $\preccurlyeq$ well-orders H, there exists an i

such that if $j \geq i$, $f(x_j, \ell_j) = f(x_i, \ell_i)$. Let $\langle y,u \rangle = f(x_i, \ell_i)$. Then $y \leq x_i$. By definition of R we have that if $j > i$, $\ell_j \in p_2(R_{\langle y,u \rangle}(x_{j+1}, \ell_{j+1}))$. That is, $\beta = \ell_{i+1}, \ell_{i+2}, \ldots$ is a thread of $\gamma = R_{\langle y,u \rangle}(x_{i+1}), R_{\langle y,u \rangle}(x_{i+2}), \ldots$. Since R and $R_{\langle y,u \rangle}$ are well-formed, the S-sequence associated with β for $R(x_{i+1})$, $R(x_{i+2}) \ldots$ is the same as the S-sequence associated with β for γ. so it is sufficient to show that <u>this</u> S-sequence is of type Ω.

Let $\pi_y = \pi \cap T_y$. We know that the S-sequence associated with a thread of $(R_{\langle y,u \rangle}|\pi_y)_\omega$ is of type Ω. We want to be able to say that the ℓ_{i+1} element of $R_{\langle y,u \rangle}(x_{i+1})$ can be traced back to the first and only element of $R_{\langle y,u \rangle}(y)$, that is, that there exists a finite string of numbers β' such that $\beta' \cdot \beta$ is a thread of $(R_{\langle y,u \rangle}|\pi_y)_\omega$. Let $z \cdot \sigma = x_{i+1}$ where $z \in T_y$ and $\sigma \in \{0,1\}$. Since $f(x_{i+1}, \ell_{i+1}) = \langle y,u \rangle$, there must exist a q such that $f(z,q) = \langle y,u \rangle$ and $q \in p_2(R(x_{i+1}, \ell_{i+1}))$. So $g(z,q) \in p_2(R_{\langle y,u \rangle}(x_{i+1}, \ell_{i+1}))$. If z=y we are done. Otherwise, $g(z,q) = q$, so we have that $q \cdot \beta$ is a thread of $R_{\langle y,u \rangle}(z) \cdot \gamma$ and $f(z,q) = \langle y,u \rangle$. Continuing in this way we see that there exists β' such that $\beta' \cdot \beta$ is a thread of $(R_{\langle y,u \rangle}|\pi_y)_\omega$. So R is accepting. □

Section 4: Proof of Theorem 6

<u>Definition:</u> Let $\theta : A \to B$, let $\Omega = (\langle L_i, U_i \rangle)_{1 \leq i \leq k}$ be a sequence of pairs of subsets of B. Then we say θ <u>is of type Ω-empty</u>, written $\theta \in [\Omega, e]$, if $\theta \in [\Omega]$ and for <u>some</u> i, $1 \leq i \leq k$, $\theta(A) \cap L_i = \phi$ (where $\theta(A) = \{\theta(a) \mid a \in A\}$).

The point of the definition will be that the property of a sequence being of type Ω-empty is basically simpler than the property of being of type Ω. In particular, if Ω is of order k we can "recognize" if a

sequence is of type Ω-empty with an Ω' of order k-empty. This is made more precise in Lemma 9.

Now let k and n be positive integers such that if $\mathcal{A}$ is a pairs-automaton of order k-empty, then $D(\mathcal{A}) \subseteq T(\mathfrak{M}^n_{\mathcal{A}})$. Let $\mathcal{A} = \langle S, \Sigma, M, s_0, \Omega \rangle$ be a pairs-automaton where $S = \{s_0, s_1, \ldots, s_{m-1}\}$ and $\Omega = (\langle L_i, U_i \rangle)_{1 \le i \le k}$. Let $t=(v,T)$ be a Σ-tree such that $t \in D(\mathcal{A})$. Eventually we will show that $t \in T(\mathfrak{M}^{n2^k}_{\mathcal{A}})$.

Below, we will define a pairs-automaton of order k-empty whose state set is $S \times P([k])$. To this end,

Definition: If $\alpha \in S^\omega$, $\alpha = u_0, u_1, \ldots$, define $\alpha^\Omega \in (S \times P([k]))^\omega$ by $\alpha^\Omega = \langle u_0, N_0 \rangle, \langle u_1, N_1 \rangle, \ldots$ where $N_0 = \emptyset$ and $N_{i+1} = N_i \cup \{j \mid u_i \in L_j\}$ for all $i \in \omega$.

Notation: If N is a finite set of positive integers and j is a positive integer, let N(j) be the j^{th} smallest member of N (if $c(N) \ge j$; otherwise N(j) is undefined). That is, N(j) is $\gamma(j)$ where γ is the finite sequence obtained by listing N in increasing order.

Definition: Let $\Omega' = (\langle L'_i, U'_i \rangle)_{1 \le i \le k}$ where for $1 \le i \le k-1$ we let $U'_i = \{ \langle u, N \rangle \in S \times P([k]) \mid i \le c(N) \le k-1 \text{ and } u \in U_{N(i)} \}$ and $L'_i = \{ \langle u, N \rangle \in S \times P([k]) \mid i \le c(N) \le k-1 \text{ and } u \in L_{N(i)} \}$; define $U'_k = \{ \langle u, N \rangle \in S \times P([k]) \mid \text{for some } j,\ 1 \le j \le k,\ u \in U_j \text{ and } j \notin N \text{ (so } c(N) < k) \}$ and $L'_k = \emptyset$. Note that Ω' is of order k-empty.

Lemma 9: Let $\alpha \in S^\omega$. Then $\alpha \in [\Omega, e] \Leftrightarrow \alpha^\Omega \in [\Omega']$.

Proof of Lemma 9: Let $\alpha = u_0, u_1, \ldots$; let $\alpha^\Omega = \langle u_0, N_0 \rangle, \langle u_1, N_1 \rangle, \ldots$. Let $N = \{j,\ 1 \le j \le k \mid \text{for some } i \in \omega,\ u_i \in L_j\}$. Then by the definition of α^Ω, we have $In(\alpha^\Omega) = \{ \langle u, N \rangle \mid u \in In(\alpha) \}$.

$\Rightarrow$: Let $\alpha \in [\Omega, e]$. Then $c(N) < k$. Let j be such that $In(\alpha) \cap U_j \neq \emptyset$

and $In(\alpha) \cap L_j = \phi$. If $j \notin N$, then $In(\alpha^{\Omega}) \cap U'_k \neq \phi$ so $\alpha^{\Omega} \in [\Omega']$. If $j \in N$, there exists q such that $N(q)=j$. Then $In(\alpha^{\Omega}) \cap U'_q \neq \phi$ and $In(\alpha^{\Omega}) \cap L'_q = \phi$. So $\alpha^{\Omega} \in [\Omega']$.

$\Leftarrow$: Let $\alpha^{\Omega} \in [\Omega']$. If $In(\alpha^{\Omega}) \cap U'_k \neq \phi$, then there exists j such that $In(\alpha) \cap U_j \neq \phi$ and $j \notin N$. So $\alpha \in [\Omega, e]$. If there exists q, $1 \leq q \leq k-1$ such that $In(\alpha^{\Omega}) \cap U'_q \neq \phi$ and $In(\alpha^{\Omega}) \cap L'_q = \phi$, then $In(\alpha) \cap U_{N(q)} \neq \phi$ and $In(\alpha) \cap L_{N(q)} = \phi$. Since $c(N) < k$, we have $\alpha \in [\Omega, e]$. □

Definition: Define the pairs-automaton

$\mathcal{L} = \langle S \times P([k]), \Sigma, M', \langle s_0, \phi \rangle, \Omega' \rangle$ where for $s \in S$, $N \subseteq [k]$, $a \in \Sigma$,

$M'(\langle s, N \rangle, a) = \{ \langle \langle u_0, N' \rangle, \langle u_1, N' \rangle \rangle \mid \langle u_0, u_1 \rangle \in M(s,a)$ and $N' = N \cup \{j \mid s \in L_j\}\}$.

Fact 3: For $x \in T$, $s \in S$, if r is a run of $\mathcal{L}_{\langle s, \phi \rangle}$ on t_x, then $p_1 r$ is a run of $\mathcal{A}_s$ on t_x, and if $\pi \subset T_x$ is a path, then by Lemma 9,

$r|\pi \in [\Omega'] \Leftrightarrow p_1 r|\pi \in [\Omega, e]$.

(Recall that $t \in D(\mathcal{A})$.)

It would be nice if $t \in D(\mathcal{L})$. For then, since $\mathcal{L}$ is of order k-empty, we would have that $t \in T(\mathfrak{M}^n_{\mathcal{L}})$ and an accepting run of $\mathfrak{M}^n_{\mathcal{L}}$ on t would clearly yield an accepting run of $\mathfrak{M}^{n2^k}_{\mathcal{A}}$ on t. It is not however necessarily true that $t \in D(\mathcal{L})$. But what if we had a set

$\mathcal{H}_{\langle \Lambda, s_0 \rangle} \subseteq T \times (S \times P([k]))$ such that $t\mathcal{H}_{\langle \Lambda, s_0 \rangle} \in D(\mathcal{L})$? . . . . .

With this approach in mind we define a (possibly trans-finite) sequence of pairwise disjoint subsets of $T \times S$, $\{H^{\delta} \mid \delta < \gamma\}$ for some ordinal γ, as follows:

Let $H^0 = \phi$. Assume that H^{δ} has been defined for $\delta < \beta$. Define

$$H^{\beta} = \left\{ \langle x, s \rangle \;\middle|\; \begin{array}{l} \text{for all } r \in Rn(\mathcal{A}_s, t_x), \text{ either} \\ \text{there exists a path } \pi \subset T \text{ such that } r|\pi \in [\Omega, e] \quad \text{or} \\ \text{there exists } y > x \text{ such that } \langle y, r(y) \rangle \in \bigcup_{\delta < \beta} H^{\delta}. \end{array} \right\} - \bigcup_{\delta < \beta} H^{\delta}.$$

Let γ be the least ordinal > 0 such that $H^{\gamma} = \phi$. Let $H = \bigcup_{\delta < \gamma} H^{\delta}$.

Lemma 10: $\langle \Lambda, s_0 \rangle \in H$.

Proof of Lemma 10: Assume that $\langle \Lambda, s_0 \rangle \notin H$. We will construct by induction a run r of $\mathcal{A}$ on t such that for all paths $\pi \subset T$, $r|\pi \notin [\Omega]$, contradicting the fact that $t \in D(\mathcal{A})$.

As in the proof of Lemma 8, if $Y \subset T$ is a set of (pairwise) incomparable points, define $T^Y = \{z \in T | \text{for all } y \in Y,\ z \not\geq y\}$.

Stage 0: Let $Y_0 = \{\Lambda\}$ and define $r(\Lambda) = s_0$. Clearly Y_0 is a set of incomparable nodes and for $y \in Y_0$, $\langle y, r(y) \rangle \notin H$.

Stage i+1: Assume that $Y_i \subset T$ is a set of incomparable nodes and that r has been defined on T^{Y_i} such that if $y \in Y_i$, then $\langle y, r(y) \rangle \notin H$.

Let $y \in Y_i$, $r(y) = u$. Since $\langle y, u \rangle \notin H$, there exists $r_y \in Rn(\mathcal{A}_u, t_y)$ such that for all paths $\pi \subset T_y$, $r_y|\pi \notin [\Omega, e]$ and such that for $x > y$, $\langle x, r_y(x) \rangle \notin H$. Let

$$G_y = \left\{ x \in T_y \;\middle|\; \begin{array}{l} \text{I) } x > y \\ \text{II) for all } j,\ 1 \leq j \leq k, \text{ there exists } z,\ y \leq z \leq x, \text{ such that } r_y(z) \in L_j \\ \text{III) } x \text{ is a minimal (under } \leq) \text{ node satisfying I) and II)} \end{array} \right\}.$$

Clearly G_y is a set of incomparable points.

Let $Y_{i+1} = \bigcup_{y \in Y_i} G_y$. Clearly Y_{i+1} is a set of incomparable points and $T^{Y_{i+1}} \supseteq T^{Y_i}$. Let $z \in T^{Y_{i+1}} - T^{Y_i}$. There exists a unique $y \in Y_i$ such that $y < z$. Define $r(z) = r_y(z)$. So if $x \in Y_{i+1}$, $\langle x, r(x) \rangle \notin H$.

This completes the definition of r. Since for all i, every member of Y_i is of length at least i, r is defined on all of T. Clearly r is a run of $\mathcal{A}$ on t. Let $\pi \subset T$ be a path. Let $Y = \bigcup_{i \in \omega} Y_i$.

Case 1: $c(\pi \cap Y)=\omega$.

Then if $i \in \omega$ there exists $y_i \in Y_i$ such that $y_i \in \pi$, and $y_0 < y_1 < \ldots$. So for every $i \in \omega$ and every j, $1 \leq j \leq k$, we have $r([y_i, y_{i+1}]) \cap L_j \neq \emptyset$. So for every j, $1 \leq j \leq k$, $In(r|\pi) \cap L_j \neq \emptyset$. So $r|\pi \notin [\Omega]$.

Case 2: $\pi \cap Y$ is finite.

Let $y = \underset{\text{under } \leq}{\text{maximum}}(\pi \cap Y)$. Let $\pi_y = \pi \cap T_y$. Then $r|\pi_y = r_y|\pi_y$. Also, it must be the case that for some j, $1 \leq j \leq k$, $r_y(\pi_y) \cap L_j = \emptyset$. But r_y was chosen so that $r_y|\pi_y \notin [\Omega, e]$. So $r_y|\pi_y \notin [\Omega]$. So $r|\pi \notin [\Omega]$.

Contradiction. □

Definition: For $\delta < \gamma$, and $<x,s> \in H^\delta$, define $H_{<x,s>} = \bigcup_{\beta<\delta} H^\beta$; let $\mathcal{H}_{<x,s>} = \{<y, <u, N>> \in T \times (S \times P([k])) \mid <y,u> \in H_{<x,s>}\}$.

By the definition of H, $<x,s> \in H$ implies that for all $r \in Rn(\mathcal{U}_s, t_x)$ either there exists a path $\pi \subset T$ such that $r|\pi \in [\Omega, e]$, or there exists $y > x$ such that $<y, r(y)> \in H_{<x,s>}$. Therefore, by Fact 3, we know that if $<x,s> \in H$, then $t_x \mathcal{H}_{<x,s>} \in D(\overline{\mathcal{X}}_{<s,\phi>})$. By the hypothesis of Theorem 6, $D(\overline{\mathcal{X}}_{<s,\phi>}) \subseteq T(\mathfrak{M}^n_{\overline{\mathcal{X}}<s,\phi>})$. So let $R'_{<x,s>}$ be an accepting run of $\mathfrak{M}^n_{\overline{\mathcal{X}}<s,\phi>}$ on $t_x \mathcal{H}_{<x,s>}$ for $<x,s> \in H$. We would like $R'_{<x,s>}$ to have the following property: For all $y \in T_x$, $\sigma \in \{0,1\}$, ℓ, q, if $<u_1, N_1> = p_1(R'_{<x,s>}(y, \ell))$ and $<u_2, N_2> = p_1(R'_{<x,s>}(y^\frown\sigma, q))$ and $\ell \in p_2(R'_{<x,s>}(y^\frown\sigma, q))$, then $N_2 = N_1 \cup \{j \mid u_1 \in L_j\}$. If this property does not hold we could remove ℓ from $p_2(R'_{<x,s>}(y^\frown\sigma, q))$ and what we'd have left would still be an accepting run of $\mathfrak{M}^n_{\overline{\mathcal{X}}<s,\phi>}$ on $t_x \mathcal{H}_{<x,s>}$. So for $<x,s> \in H$, assume without loss of generality that $R'_{<x,s>}$ has this property.

For $\langle x,s\rangle \in H$, define $R_{\langle x,s\rangle}: T_x \to S^{n2^k}_{\mathcal{U}}$ as follows: if $y \in T_x$, let $R_{\langle x,s\rangle}(y)$ be the same length as $R'_{\langle x,s\rangle}(y)$; if $1 \le i \le \text{length}(R'_{\langle x,s\rangle}(y))$, define $p_2(R_{\langle x,s\rangle}(y,i)) = p_2(R'_{\langle x,s\rangle}(y,i))$ and define $p_1(R_{\langle x,s\rangle}(y,i)) = p_1(p_1(R'_{\langle x,s\rangle}(y,i)))$. So $R_{\langle x,s\rangle} \in \text{Rn}(\mathfrak{M}^{n2^k}_{\pi_s}, t_x H_{\langle x,s\rangle})$. Let $\pi \subset T_x$ be a path; let β be a thread of $(R_{\langle x,s\rangle}|\pi)_\omega$ and let α be the S-sequence associated with β for $(R_{\langle x,s\rangle}|\pi)_\omega$. Then β is a thread of $(R'_{\langle x,s\rangle}|\pi)_\omega$, and the S-sequence associated with β for $(R'_{\langle x,s\rangle}|\pi)_\omega$ is, by the above property, α^{Ω}. We know, since $R'_{\langle x,s\rangle}$ is an accepting run of $\mathfrak{M}^n_{\mathcal{U}_{\langle s,\phi\rangle}}$, that $\alpha^{\Omega} \in [\Omega']$. So by Lemma 9, $\alpha \in [\Omega, e]$, so $\alpha \in [\Omega]$.

By the above paragraph and Lemma 7, we can finally conclude that for $\langle x,s\rangle \in H$, there exists a well-formed accepting run of $\mathfrak{M}^{n2^k}_{\pi_s}$ on $t_x H_{\langle x,s\rangle}$. Denote it by $R_{\langle x,s\rangle}$.

We now proceed to construct R, a well-formed run of $\mathfrak{M}^{n2^k}_{\pi}$ on t. We first define a total ordering, $\preccurlyeq_H$, on H as follows: Let $\langle x,s\rangle \in H^{\delta}$ and $\langle y,u\rangle \in H^{\beta}$; then we put $\langle y,u\rangle \preccurlyeq_H \langle x,s\rangle$ if $\beta < \delta$, or if $\beta = \delta$ and $\langle y,u\rangle \preccurlyeq \langle x,s\rangle$ where $\preccurlyeq$ is a fixed well-ordering of $T \times S$ as in Section 3. Clearly $\preccurlyeq_H$ is a well-ordering of H. Denote by $\prec_H$ the obvious strict well-ordering determined by $\preccurlyeq_H$. It is important to note that if $\langle y,u\rangle \in H_{\langle x,s\rangle}$, then $\langle y,u\rangle \prec_H \langle x,s\rangle$.

At the same time as we define R we will define a function $f: \{\langle \Lambda, 1\rangle\} \cup ((T-\{\Lambda\}) \times [mn2^k]) \to H$. As before, we carry along the following induction hypothesis: If R(x) is defined and $1 \le \ell \le \text{length}(R(x))$, then $f(x,\ell)$ is defined; if $f(x,\ell) = \langle y,u\rangle$ and $p_1(R(x,\ell)) = s$, then

a) $y \le x$ and

b) if $y = x$ then $u = s$ and

c) $\langle x,s\rangle \notin H_{\langle y,u\rangle}$.

Let $R(\Lambda)=<s_0,\emptyset>$. Let $f(\Lambda,1)=<\Lambda,s_0>$. Clearly the above hypothesis holds so far.

Assume now that R(x) and f have been defined so that the above hypothesis holds. For ℓ, $1\leq\ell\leq$ length(R(x)), define

$$g(x,\ell)=\begin{cases}1 \text{ if } p_1(f(x,\ell))=x\\ \ell \text{ if } p_1(f(x,\ell))\neq x\end{cases}.$$

Let $\sigma\in\{0,1\}$. Define $R(x^\frown\sigma)$ to be of length $mn2^k$ such that for all q, $1\leq q\leq mn2^k$, $p_1(R(x^\frown\sigma,q))=s_i$ where i=the greatest integer less than $\frac{q}{n2^k}$, and such that $p_2(R(x^\frown\sigma,q))=$

$\{\ell \mid f(x,\ell)$ is defined and $g(x,\ell)\in p_2(R_{f(x,\ell)}(x^\frown\sigma,q))\}$.

We now define f. Let $s=p_1(R(x^\frown\sigma,q))$. If $<x^\frown\sigma,s>\notin H$, define $f(x^\frown\sigma,q)=$minimum under $\preccurlyeq_H$ $\{f(x,\ell)\mid \ell\in p_2(R(x^\frown\sigma,q))\}$. If $<x^\frown\sigma,s>\in H$, define $f(x^\frown\sigma,q)=$minimum under $\preccurlyeq_H$ $(\{<x^\frown\sigma,s>\}\cup\{f(x,\ell)\mid \ell\in p_2(R(x^\frown\sigma,q))\})$. Clearly a) and b) are preserved. To see that c) is true at $x^\frown\sigma$, observe that if $<x^\frown\sigma,s>\in H_{f(x^\frown\sigma,q)}$ then $<x^\frown\sigma;s>\in H$ and $<x^\frown\sigma,s><_H f(x^\frown\sigma,q)$, contradicting the definition of $f(x^\frown\sigma,q)$.

This completes the definition of R and f. That $R\in Rn(\mathfrak{M}^{n2^k}_{\sigma\ell},t)$ follows exactly as in the previous section. It remains to show that R is accepting. So let $\pi\subset T$ be a path, $\pi=\{x_0,x_1,\ldots\}$ where $x_i<x_{i+1}$ for all $i\in\omega$. Let $\ell_0,\ell_1,\ldots$ be a thread of $(R|\pi)_\omega$. Now for all $i\in\omega$, $f(x_{i+1},\ell_{i+1})\preccurlyeq_H f(x_i,\ell_i)$. So, since $\preccurlyeq_H$ well-orders H, there exists an i such that $j\geq i$ implies that $f(x_j,\ell_j)=f(x_i,\ell_i)$. Therefore the proof that R is accepting is identical to that in Case 2) at the end of Section 3. □

Proof of Theorem 3:

Notation: For k a nonnegative integer, let $n(k)=2^{\frac{k(k+1)}{2}}$. Note that $2^{k+1}n(k)=n(k+1)$.

Let our induction hypothesis at stage at stage K be that if $\mathcal{A}$ is a pairs-automaton of order K, then $D(\mathcal{A}) \subseteq T(\mathfrak{M}_{\mathcal{A}}^{n(K)})$.

The hypothesis is true at K=0 by Theorem 4. Assume it is true for K=k. We wish to show it for K=k+1. By Theorem 5, if $\mathcal{B}$ is a pairs-automaton of order k+1-empty, then $D(\mathcal{B}) \subseteq T(\mathfrak{M}_{\mathcal{B}}^{n(k)})$. Therefore, by Theorem 6, if $\mathcal{A}$ is of order k+1, $D(\mathcal{A}) \subseteq T(\mathfrak{M}_{\mathcal{A}}^{n(k+1)})$.

This together with Lemma 6 completes the proof. □

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13. ABSTRACT

In [6] Rabin defines Automata on Infinite Trees, and the body of that paper is concerned with proving two theorems about these automata.

The result we consider in the first chapter says that there exists an effective procedure to determine, given an automaton on infinite trees, whether or not it accepts anything at all. We present an alternative proof which reduces the emptiness problem for automata on infinite trees to that for automata on finite trees. This proof is much simpler than Rabin's, and has as corollaries other results which he proves in [5].

Chapter 2 is concerned with the more difficult result of [6] that for every automaton on infinite trees, $\mathcal{A}$, there exists another one, $\mathcal{A}'$, such that $\mathcal{A}'$ accepts precisely the complement of the set accepted by $\mathcal{A}$. Rabin's construction of $\mathcal{A}'$ and the proof that it works is an involved induction. In this paper we present a fairly simple description of a complement machine $\mathcal{A}'$, given $\mathcal{A}$, such that it is very plausible that $\mathcal{A}'$ works in the sense that $T(\mathcal{A}')=\overline{T(\mathcal{A})}$. The proof that our construction works, however, is difficult and very similar in complexity to Rabin's proof in [6] that his (more difficult) construction works.

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