UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE M19 - 232

By Gene H. Golub

TITLE — Solution of the Matrix Equation $Ax = \lambda Bx$ Where A and B are Symmetric and B Is Positive Definite.

TYPE — Entire Program.

ACCURACY — Depends on condition of B.

DURATION

(a) 33 seconds to input routine

(b) $(10I + 2.1)\, n^3 + 5.2\, n^2$ milliseconds where n = order of matrix and I = number of iterations. I varies from 4 for n = 3 to 7 for n = 19.

CAPACITY — Matrices of order 19.

METHOD OF USE

(1) Read program tape into memory in usual way. If no reading error has been committed machine will stop on $24\ (447)_{16}$

(2) Place data tape in reader and restart machine with black switch.

(3) After computation has been completed, results will be punched out for printing.

(4) The machine stops on $24\ (047)_{16}$. A new problem can be begun by repeating step 2.

PUNCHING OF THE DATA — To compute the eigenvalues or eigenvectors of the matrix equation $Ax_i = \lambda_i Bx_i$ $(i = 1,2,\ldots,n)$ where A and B are symmetric and B is positive definite proceed as follows:

(a) Scale the matrices so that

$$\sum_{i,j=1}^{n} a_{ij}^2 < 1/2 \quad \text{and} \quad \sum_{i,j=1}^{n} b_{ij}^2 < 1/2$$

(b) Punch the lower half of the matrix A,

$$a_{11}$$

$$a_{21} \quad a_{22}$$

$$a_{31} \quad a_{32} \quad a_{33}$$

$$a_{n1} \quad a_{n2} \quad \cdots\cdots \quad a_{nn}$$

row by row, as a sign followed by up to twelve decimal digits. The last element a_{nn} is followed by N.

(c) Punch the matrix B in a similar fashion but follow the last element b_{nn} by an N,J,F, or L.

Let $By_i = \delta_i y_i \quad i = 1,2,\ldots,n.$

Then the results of punching these characters is as follows:

N : λ_i , $(i = 1,\ldots,n)$ are punched

J : λ_i, x_i $(i = 1,\ldots,n)$ are punched

F : δ_i, λ_i $(i = 1,\ldots,n)$ are punched

L : δ_i, λ_i, x_i $(i=1,\ldots,n)$ are punched.

(d) The last character is followed by a sexadecimal character p which determines the number of decimal digits to be printed. The character p can assume the values 1,2,3,... 9,K,S,N where K = 10, S = 11, and N = 12.

THE PRINTED RESULTS First, the eigenvalues of B are punched if desired. Next if only the eigenvalues of the system $Ax = \lambda x$ are computed then they are punched in a single column. If the eigenvalues and eigenvectors are computed, then following each λ_i will be the corresponding vector x_i.

Throughout this program, there are a number of checks. They are as follows:

FF 02S	Matrix A fails to have $\frac{n^2+n}{2}$ elements
FF 02N	Matrix B fails to have $\frac{n^2+n}{2}$ elements
FF 02J	A and B fail to have same number of elements
FF 02F	At least one eigenvalue of B $\leq 5\text{x}10^{-10}$

DISCUSSION OF METHOD USED

To solve the equation $|A - \lambda B| = 0$ we first solve $|B - \delta I| = 0$. Since B is symmetric we know that an orthogonal matrix Y and a diagonal matrix Δ exists such that $B = Y\Delta Y^T$; Y is the matrix of eigenvectors and Δ the matrix of eigenvalues for B. The

elements of Δ are non-negative since B is positive definite, so we may write

$$B = Y\Delta^{1/2}\ \Delta^{1/2}\ Y^T = UU^T$$

where $U = Y\Delta^{1/2}$. Hence we have

$$|A - \lambda B| = |A - \lambda UU^T| = 0$$

and

$$|U^{-1}AU^{-T} - \lambda I| = 0.$$

The matrix $U^{-1}AU^{-T}$ is symmetric so that there is an orthogonal matrix Z and a diagonal matrix Λ such that

$$U^{-1}\ AU^{-T} = Z\Lambda Z^T.$$

The eigenvalues λ_i are, then, the elements of Λ. By definition of an eigenvector, we have

$$(U^{-1}\ AU^{-T})\ z_i = \lambda_i z_i$$

$$= \lambda_i\ U^T\ U^{-T}\ z_i$$

and $AU^{-T}z_i = \lambda_i\ UU^T\ U^{-T}\ z_i = \lambda_i\ BU^{-T}z_i$

Then if we let $x_i = U^{-T}z_i$ we have $Ax_i = \lambda_i Bx_i$

so that λ_i and x_i are the eigenvalues and eigenvectors of the system $|A - \lambda B| = 0$.

The matrix multiplication $U^{-T}Z$ is performed by replacing the identity matrix by U^{-T} when finding the eigenvalues of $U^{-1}AU^{-T}$. Therefore, $x_i^T\ Bx_i = 1$ for $i = 1,2,\ldots,n$.

NOTE

Frequently, it is of interest to find the eigenvalues and eigenvectors of Hermitian matrices. The present routine can handle these problems, although in a somewhat wasetful manner.

Suppose $A = G + i\ H$

and $B = M + i\ N$

Then if B is positive definite,

$$(G + i\,H)(x + iy) = \lambda(M + iN)(x+iy)$$

$$Gx - Hy = \lambda(Mx - Ny)$$

$$Hx + Gy = \lambda(Nx + My)$$

That is

$$\begin{bmatrix} G, & -H \\ H, & G \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} M, & -N \\ N, & M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

But $-H = H^T$ and $-N = N^T$

Therefore

$$\begin{bmatrix} G, & H^T \\ H, & G \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} M, & N^T \\ N & M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Solving this equation will yield 2n roots. Each root of the original system will be duplicated.

DATE February 13, 1957
CODED BY
APPROVED BY D. E. Muller

LOCATION	ORDER	NOTES PAGE 1
	00 3K	
0	00 F	
	00 473F	
1	00 F	
	00 F	
2	00 F	
	00 20F	Location of parameters
3	00 F	
	00 190F	Location of P-16
4	00 F	
	00 253F	Location of R-1
5	00 F	
	00 30F	Location of Y-1
6	00 F	Location of matrix A on drum
	00 3560F	
7	00 F	
	00 302F	
8	00 F	Location of subroutines on drum
	00 2560F	
9	00 F	Location of M-4
	00 110F	
	00 20K	
0	00 F	
	00 F	
1	00 (p)F	
	00 F	
2	80 F	
	00 (n)F	
3	00 F	
	00 F	
4	00 F	
	00 F	
5	80 F	
	00 F	
6	00 (n)F	
	00 (n)F	

LOCATION	ORDER	NOTES	PAGE 2
7	00 F		
	00 500J		
8	00 1F		
	00 1F		
9	80 F		
	00 (n^2)F		
	00 30K		
Library Routine Y-1 (modified)			
	00 71K		
0	50 110F	from 35	
	50 L		
1	26 S8		Read input routine off drum
	00 SS		
2	00 73F		
	50 2L		
3	26 110F		Enter input routine
	L5 S5		
4	80 1F		
	L4 36L		
5	40 27L		
	L5 6S5		
6	46 10L		
	46 27L		
7	50 SN		Read M-4 off drum
	50 7L		
8	26 S8		
	00 100SS		
9	00 165F		
	00 1F		
10	50 ( )F	by 6	
	50 10L		Enter M-4
11	26 SN		
	00 1F		
12	50 110F		Read in routine II off drum
	50 12L		

LOCATION	ORDER	NOTES	
13	26 S8		
	00 300SS		
14	00 45F		
	00 1F		
15	50 S6		Read in P-16 off drum
	50 15L		
16	26 S8		
	00 350SS		
17	00 56F		
	50 17L		Enter routine II
18	26 110F		
	00 1F		
19	50 110F		Read routine III from drum
	50 19L		
20	26 S8		
	00 425SS		
21	00 70F		
	50 21L		Enter routine III
22	26 110F		
	00 1F		
23	50 SN		
	50 23L		Read M-4 from drum
24	26 S8		
	00 100SS		
25	00 165F		
	L5 37L		Modify M-4
26	40 8SN		
	00 1F		
27	50 ( )F	by 5,6	
	50 27L		Enter M-4
28	26 SN		
	00 1F		
29	50 110F		Read routine IV from drum
	50 29L		
30	26 S8		
	00 500SS		

LOCATION	ORDER	NOTES	PAGE 4
31	00 43F		
	00 1F		
32	50 S6		Read P-16 from drum
	50 32L		
33	26 S8		
	00 350SS		
34	00 56F		
	50 34L		Enter routine IV
35	26 110F		
	24 L		
36	J0 F		
	50 27L		
37	L5 F		
	32 130F		
Input Routine			
	00 110K		
0	K5 F		
	42 38L		Plant link
1	50 S3		
	50 1L		Input matrix A
2	26 41L		
	L3 4F		
3	36 4L		
	FF 043F		
4	L5 3F		
	40 8F		
5	L0 8S5		
	42 2S5		
6	42 6S5		Compute parameters
	00 20F		
7	46 6S5		
	50 6S5		
8	75 6S5		
	10 1F		

LOCATION	ORDER	NOTES	PAGE 5
9	42 9S5		
	00 20F		
10	46 30L		
	L4 6S5		
11	10 21F		
	F4 30L		
12	42 4S5		
	L5 30L		
13	42 15L		
	41 0F		
14	F5 15L	from 19	
	42 15L		
15	00 1F		
	L5 F	by 13,14,23; from 25	
16	32 16L		
	40 190S3	by 17	
17	F5 16L		
	40 16L		square matrix A
18	F5 0F		
	40 0F		
19	L0 40L		
	36 14L		
20	40 1F		
	L7 1F		
21	32 22L		
	L5 15L		
22	42 30L		
	L5 15L	from 21	
23	L4 0F		
	42 15L		
24	L5 0F		
	L0 2S5		
25	32 15L		
	F5 40L		
26	40 40L		
	F0 2S5		

LOCATION	ORDER	NOTES	
27	36 28L		
	22 12L		
28	J0 190S3		
	50 28L		Record A on drum
29	26 S8		
	00 S9		
30	00 ( )F		
	50 1023S3	by 22	
31	50 S3		
	50 31L		Input matrix B
32	26 41L		
	00 38F		
33	40 S5		
	81 4F		
34	00 20F		
	46 -1S5		
35	L3 4F		
	32 36L		
36	FF 044F		
	L5 8F		
37	L0 3F		
	40 0F		
38	L3 F		
	32 ( )F	by 0	
39	FF 045F		
	00 F		
40	80 F		
	00 1F	by 26	
41	50 42L		
	26 999F		
42	00 F		
	00 41L		
43	00 F		
	26 41L		
	26 1N		

LOCATION	ORDER	NOTES	PAGE 7
Modified N-2			
	00 800K		
0	J0 110F		
	50 L		
1	26 S8		
	00 SS		Place input routine on drum
2	00 73F		
	L5 40L		
3	40 500F		
	26 999F		
	26 800N		
Routine II			
	00 110K		
0	K5 F		
	42 44L		Plant link
1	L5 S5		Print eigenvalues of B ?
	36 4L		
2	92 135F		
	92 515F		
3	L5 1S5		Plant print parameter
	46 13L		
4	41 3F	from 1	
	L5 5S5		
5	40 4F		
	L5 S3	by 17; from 19	
6	40 5F		Store eigenvalues
	40 S3	by 16	of B consecutively
7	L0 7S5		
	36 9L		Are eigenvalues of B $\geq 5 \times 10^{-10}$?
8	FF 046F		
	24 000F		
9	L4 7S5	from 7	
	L0 4F		
10	32 11L		Test for smallest eigenvalue
	L5 5F		

LOCATION	ORDER	NOTES	PAGE 8
11	40 4F		
	L5 S5	from 10	
12	32 15L		
	L5 5F		
13	50 ( )F	by 3	
	50 13L		
14	26 S6		
	92 131F		
15	92 515F		
	F5 6L	from 12	
16	40 6L		
	F5 5L		
17	F4 3F		
	42 5L		
18	F5 3F		
	40 3F		
19	L0 2S5		
	32 5L		
20	L5 4F		
	50 20L		
21	26 S7		
	49 3F		
22	19 38F		
	40 3S5	from 26	
23	F5 2F		
	L0 3F		
24	36 27L		
	L5 3F		
25	10 1F		
	40 3F		
26	F5 3S5		
	22 22L		
27	41 6F	from 24	
	41 0F		
28	L5 3S5		
	42 34L		

LOCATION	ORDER	NOTES	PAGE 9
29	41 8F	from 43	
	L5 4S5		
30	L4 6F		
	42 33L		
31	42 36L		
	L5 S3	by 41	
32	32 32L		
	50 32L		
33	26 S7		
	L5 ( )F	by 30,38; from 40	
34	50 0F		
	10 ( )F	by 28	
35	66 2F		
	S5 F		
36	32 36L		
	40 ( )F	by 31,38	
37	L5 33L		
	L4 6S5		
38	42 33L		
	42 36L		
39	F5 8F		
	40 8F		
40	L0 2S5		
	32 33L		
41	F5 31L		
	40 31L		
42	F5 6F		
	40 6F		
43	L0 2S5		
	36 29L		
44	00 1F		
	22 ( )F	by 0	
	00 800K		
0	J0 110F		
	50 L		Record routine II on drum.

LOCATION	ORDER	NOTES
1	26 S8	
	00 300SS	
2	00 45F	
	L5 40S8	
3	40 501F	
	26 999F	
	26 800N	
Routine III		
	00 110K	
0	K5 F	
	42 69L	
1	L5 9S5	
	00 20F	
2	46 5L	
	41 0L	
3	50 SK	by 2,6
	50 3L	
4	26 S8	
	00 S9	
5	00 F	by 2
	L5 13L	from 31
6	46 3L	
	43 14L	
7	47 14L	
	43 11L	
8	41 1L	
	L5 3L	from 23
9	46 13L	
	L5 4S5	
10	L4 1L	
	42 13L	
11	41 2L	
	41 ( )F	by 7,21
12	2L 13L	
	S5 F	from 19

LOCATION	ORDER	NOTES PAGE 11
13	50 SK	by 9,17 $\{a_{ij}\}$
	74 (u_{ij})F	by 10,16 AU^{-T}
14	L4 ( )F	by 7,20
	40 ( )F	by 6,20
15	L5 13L	
	L4 2S5	
16	42 13L	
	L4 8S5	
17	46 13L	
	F5 2L	
18	40 2L	
	L0 2S5	
19	32 12L	
	L5 14L	
20	L4 8S5	
	40 14L	
21	42 11L	
	F5 1L	
22	40 1L	
	L0 2S5	
23	32 8L	
	41 2L	
24	47 25L	
	00 1F	Waste
25	L5 ( )F	by 24,27 ; from 29
	40 SK	by 27
26	L5 25L	
	L4 8S5	
27	40 25L	
	F5 2L	
28	40 2L	
	L0 2S5	
29	36 25L	
	F5 0L	
30	40 0L	
	L0 2S5	

LOCATION	ORDER	NOTES
31	32 5L	
	F5 5S5	
32	40 3L	
	41 0L	
33	41 1L	from 39
	43 39L	
34	47 42L	
	43 42L	
35	L5 0L	from 50
	L4 4S5	
36	00 20F	
	46 41L	
37	L5 1L	
	L4 40L	
38	42 41L	
	41 2L	
39	00 1F	
	41 (0)F	by 33,48
40	2L 41L	
	S5 SK	from 46
41	50 (u_{ij})F	by 36,44 ; from 40
	74 (a_{ij})F	by 38,44 $U^{-1}(AU^{-T})$
42	L4 0F	by 34,47
	40 0F	by 34,47
43	L5 41L	
	L4 6S5	
44	42 41L	
	46 41L	
45	F5 2L	
	40 2L	
46	L0 2S5	
	32 40L	
47	L5 42L	
	L4 8S5	

LOCATION	ORDER	NOTES	
48	40 42L		
	42 39L		
49	F5 1L		
	40 1L		
50	L0 3L		
	36 35L		
51	J0 0F		
	50 51L		
52	26 S8		
	00 S9	by 54	
53	00 1F	by 56	
	F5 0L		
54	F4 52L		
	40 52L		
55	L5 53L		
	L4 8S5		
56	46 53L		
	F5 3L		
57	40 3L		
	F5 0L		
58	40 0L		
	L0 2S5		
59	36 33L		
	41 L		
60	50 S3	by 65	
	50 60L	from 69	
61	26 S8		
	00 S9	by 63	
62	00 1F	by 67	
	F5 61L		
63	F4 L		
	40 61L		
64	L5 60L		
	L4 62L		

LOCATION	ORDER	NOTES	PAGE 14
65	L0 8S5		
	46 60L		
66	L5 62L		
	L4 8S5		
67	46 62L		
	F5 L		
68	40 L		
	L0 2S5		
69	32 60L		
	22 ( )F	by 0	
	00 800K		
0	J0 110F		
	50 L		Record routine III on drum
1	26 S8		
	00 425SS		
2	00 70F		
	L5 40S8		
3	40 502F		
	26 999F		
	26 800N		
Routine IV			
Print Routine			
	00 110K		
0	K5 100F		
	42 38L		
1	L5 3S5		
	42 2L		
2	19 38F		
	00 ( )F	by 1,14	
3	40 0F		
	L5 1S5		
4	46 29L	by 15	
	92 131F		

LOCATION	ORDER	NOTES	PAGE 15
5	50 39L		00 F 00 1F
	L5 0L		
6	L4 29L	by 15	
	46 29L		
7	75 41L		00 F 00 10F
	L5 0F		
8	S0 F		
	32 5L		
9	S5 F		
	40 1F		
10	L5 0F		
	50 40L		00 F 00 F
11	66 1F		
	S5 F		
12	32 12L		
	40 4F		4: eigenvalue factor
13	L5 2L		5: eigenvector factor
	L4 3S5		
14	42 2L		
	L5 42L		
15	40 6L		
	46 4L		
16	F5 12L		
	42 12L		
17	L1 1L		
	40 1L		
18	36 2L		
	41 7F		
19	92 131F	from 27	
	92 515F		
20	50 5F		
	7J S3	by 35	
21	54 F		
	50 21L		
22	26 S6		
	L5 S5		

LOCATION	ORDER	NOTES
23	00 1F	
	32 34L	
24	92 131F	
	L5 4S5	
25	L4 7F	
	42 28L	
26	41 8F	
	00 1F	
27	92 131F	from 33
	92 515F	
28	50 4F	
	7J ( )F	by 31
29	54 F	by 4,6
	50 29L	
30	26 S6	
	L5 28L	
31	L4 2S5	
	42 28L	
32	F5 8F	
	40 8F	
33	L0 2S5	
	36 27L	
34	92 135F	
	F5 20L	
35	F4 7F	
	40 20L	
36	F5 7F	
	40 7F	
37	L0 2S5	
	36 19L	
38	92 131F	
	22 ( )F	
39	00 F	
	00 1F	
40	00 F	
	00 0F	

LOCATION	ORDER	NOTES	PAGE 17
41	00 F		
	00 10F		
42	L4 21L		
	46 21L		
	00 800K		
0	J0 110F		
	50 L		
1	26 S8		
	00 500SS		
2	00 43F		
	L5 40S8		
3	40 503F		
	26 999F		
	26 800N		
	00 110K		
Library Routine M-4			
Change Word 45			
	26 163L		
	50 6F		
Add to End of Code			
	L3 6F		
	32 110L		
	L5 15F		
	22 45L		
	00 800K		
0	J0 110F		
	50 L		
1	26 S8		
	00 100SS		
2	00 165F		
	L5 40S8		
3	40 504F		
	26 999F		

LOCATION	ORDER	NOTES	PAGE 18
	26 800N		
	00 190K		
Code P-16			
	00 800K		
0	J0 S6		
	50 L		
1	26 S8		
	00 350SS		
2	00 56F		
	L5 40S8		
3	40 505F		
	26 999F		
	26 800N		
	00 800K		
Code X - 7			
	24 71N		