UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE T5 - 157

TITLE	Sine-Cosine Routine	(D.O.I or SADOI)
TYPE	Closed	
NUMBER OF WORDS	21	
TEMPORARY STORAGE	0, 1, 2	
DURATION	9 milliseconds	
ACCURACY	2^{-38}	
DESCRIPTION	Given θ/π in A, the routine places (1/2) sin θ in A when a standard entry is used.	
REMARKS	(1) If A contains $1/2 \pm \theta/\pi$, the quantity 1/2 cos θ will be obtained.	
	(2) If θ/π overflows in A from left shifts or addition the result is unaffected because sin $(\theta + 2n\pi)$ = sin θ.	
EXAMPLES	(1) If memory location q contains θ/π, (1/2) cos θ may be obtained with the entry	

	LJ q
p	50 p
p + 1	26 -

(2) To find sin 3.82 radians when memory location q contains 0.382 and memory location r contains $10/4\pi$, we proceed as follows, ignoring the overflow in A.

p - 1	50 q
	75 r
	00 2F
p	50 p
p + 1	26 -

METHOD (1) Given θ in radians we evaluate sin $(\theta/2)$ by a Tchebyscheff polynomial expansion to the Taylor series expansion of the sine function, using Library routine KA-1.

(2) If θ is scaled so that $-1 \leq 2^{-n}\theta < 1$, we form 2^n $(2^{-n}\theta/\pi = \theta/\pi \pmod 2) = \theta'/\pi$.

(3) Using the identity sin a = sin $(\pi - a)$ we obtain θ'' with $-1/2 \leq \theta''/\pi < 1/2$ such that $\sin\theta'' = \sin\theta'$ $= \sin\theta$.

ERROR ANALYSIS (1) In word 4 we develop $-4\ (\theta/\pi)^2$. Consider $|\theta/\pi| < 1/2$. This quantity possesses up to 38 significant binary digits. We form $(\theta/\pi)^2 < 1/4$ and shift left twice, introducing thereby at most an error of 2^{-39}, since q_1 contains a significant binary digit.

(2) We next develop the polynomial:

$$1/2 \sin\theta = \theta/\pi\ [a_{13}(2\theta/\pi)^{12} + a_{11}(2\theta/\pi)^{10} + a_9(2\theta/\pi)^8 + a_7(2\theta/\pi)^6 + a_5(2\theta/\pi)^4 + a_3(2\theta/\pi)^2 + a_1]$$

This is accomplished by the recursion relation:

$$S_0 = 0$$

$$S_{i+1} = (2\theta/\pi)^2 S_i + a_j, \quad \begin{matrix}(i=0,1,2,\ldots 6)\\ (j=13,11,9,\ldots 1)\end{matrix}$$

At each one of these steps we introduce a round-off error due to the multiplication. Letting $(2\theta/\pi)^2 = x$, we obtain:

$$1/2 \sin\theta = \frac{2\theta}{\pi}[a_{13}x^6 + a_{11}x^5 + a_9x^4 + a_7x^3 + a_5x^2 + a_3x + a_1 + 2^{-40}(x^5 + x^4 + x^3 + x^2 + x + 1)\ \theta/\pi + 2^{-40}]$$

Since x cannot exceed 1 and $|\theta/\pi| < 1/2$ we have as error $3.2^{-40} + 2^{-40}$. The maximum total generated error is therefore 2^{-38}.

(3) There will also be developed a propagated error due to the inaccuracy of the quantity $4(\theta/\pi)^2$. We will admit only the error resulting from the term $a_3(2\theta/\pi)^2$ in equation (1) due to the relative small contributions propagated by the other terms.

Thus:

$a_3(2\theta/\pi)^2 = .4[4\ \theta^2/\pi^2 \pm 2^{-39}] = .4(2\theta/\pi^2 \pm (.4)2^{-39}$.

The propagated error is therefore: $(.4)2^{-39}$

(4) The total maximum error is therefore:

$2^{-38} + (.4)2^{-39} = 2 + .4)\ 2^{-39} = (2.4)2^{-39}$

(5) A sample of calculations for the sine of a number of arguments was found to be in error at most 3×10^{-12} which is less than the maximum error indicated above.

(6) Finally, we consider the case $|\theta/\pi| = 1/2$. The error was found experimentally to be less than 2^{-39}.

DATE	11/29/54 RT: 4/21/61
PROGRAMMED BY	Werner L. Frank
APPROVED BY	J. P. Nash

nj

LOCATION	ORDER		NOTES PAGE 1 T 5
	00K(T5)		
0	40 F		Store θ/π
	K5 14L		
1	42 12L		Plant link
	L9 F		$1/2 - \theta/\pi$
2	40 F		
	LS F		$1/2 - \lvert 1/2 - \theta/\pi \rvert = \theta'/\pi$
3	40 F		
	50 F		$-1/2 \leq \theta'/\pi < 1/2$
4	71 F		
	00 2F		$-4\ (\theta'/\pi)^2$
5	40 1F		
	L5 L		
6	42 7L		
	50 13L		
7	79 1F		$4(\theta'/\pi)^2\ S_i + a_j$
	L4 (14)L	6,9'	
8	40 2F		
	50 2F		
9	F5 7L		
	42 7L		
10	L0 20L		Test
	36 7L		
11	LJ 9L		Prepare round off
	74 F		$1/4 \sin \theta$
12	00 1F		$1/2 \sin \theta$
	22 ( )F	1	
13	00 F		
	00 27232J		$a_{13}/2 = .27232 \times 10^{-7}$
14	N0 F		
	00 499998202409J		$-a_{11}/2 = .1797591 \times 10^{-5}$
15	00 F		
	00 80219199J		$a_9/2 = .80219199 \times 10^{-4}$

LOCATION	ORDER	NOTES PAGE 2 T 5
16	NO F	
	00 497659123501J	$-a_7/2 = .2340876499 \times 10^{-2}$
17	00 F	
	00 398463130 04J	$a_5/2 = .39846313004 \times 10^{-1}$
18	NO F	.322982048742 RVP
	00 177017951258J	$-a_3/2 = .\cancel{785398163397}$
19	40 F	
	00.285398163397J	$a_1/2 = .785398163397$
20	L9 1F	End constant
	L4 20L	

.5)2

$-.322982048742$